所別：天文研究所碩士班 科目：應用數學

1. (15%) Define $\operatorname{sinc}(x)=\dfrac{\sin(x)}{x}$

   (i) Calculate $\lim_{x\to 0}\operatorname{sinc}(x)$.

   (ii) Calculate $\int_{-\infty}^{\infty}\operatorname{sinc}(x)\,dx$.

2. (15%)

   (i) For the $n\times n$ matrices A and B, in general $A\cdot B\neq B\cdot A$. Prove that, however, $trace(A\cdot B)=trace(B\cdot A)$.

   (ii) If λ is a eigenvalue of a 2×2 matrix A, show that λ satisfies
   $$\lambda^2-\lambda\cdot trace(A)+\det(A)=0.$$
   where $\det(A)$ is the determinant of matrix A

   (iii) A matrix P is a projection operator satisfying the condition $P^2=P$. Show that the eigenvalues of P are 0 and 1.

3. (15%) For the following vectors, please indicate which one (or ones) can be written as a gradient of scalar function(s) (that is, $\nabla\phi$) and calculate the corresponding scalar function(s)

   (i) $-\dfrac{y}{x^2+y^2}\hat{x}+\dfrac{x}{x^2+y^2}\hat{y}+0\hat{z}$

   (ii) $yz\hat{x}+xz\hat{y}+xy\hat{z}$

   (iii) $2xyz^2\hat{x}+\left[x^2z^2+z\cdot\cos(yz)\right]\hat{y}+\left[y\cdot\cos(yz)\right]\hat{z}$

   (iv) $x^2\hat{x}+z^2\hat{y}+y^2\hat{z}$

   where $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the unit vectors of x, y and z axes respectively.

4. (10%)

   (i) Expand the following periodic function as a Fourier series:
   $$\begin{cases} f(x)=x^2, & -\pi<x<\pi \\ f(x+2n\pi)=f(x), & n\in Z \end{cases}$$

   (ii) From the result above, calculate
   $$\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

注意：背面有試題

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5. (10%) Suppose $F(\omega)$ and $G(\omega)$ are the Fourier transformation of $f(t)$ and $g(t)$ respectively.

(i) Show that the Fourier transformation of $\int_{-\infty}^{\infty} f(\tau)\cdot g(t-\tau)\cdot d\tau$ is $F(\omega)\cdot G(\omega)$.

(ii) Show that $\int_{-\infty}^{\infty} F(\omega)\cdot[G(\omega)]^{*}\cdot d\omega = \int_{-\infty}^{\infty} f(t)\cdot[g(t)]^{*}\cdot dt$.

6. (15%) Find the general solutions for the following differential equations

(i) $x^2\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}-4y=x^3$.

(ii) $\begin{cases}\dfrac{dy}{dx}=-2y+z\\ \dfrac{dz}{dx}=-4y+3z+10\cos(x)\end{cases}$

7. (10%) If n is a positive integer, show that

(i) $\int_0^{\infty} x^{2n+1}\exp(-ax^2)\cdot dx=\dfrac{n!}{2a^{n+1}}$,

(ii) $\int_0^{\infty} x^{2n}\exp(-ax^2)\cdot dx=\dfrac{(2n-1)!!}{2^{n+1}a^n}\sqrt{\dfrac{\pi}{a}}$.

8. (10%) For the Poisson distribution $p_a(x)=e^{-a}\dfrac{a^x}{x!}$ where $a>0$, x is an integer and $x\ge 0$, calculate

(i) the mean: $\langle x\rangle \equiv \sum_{x=0}^{\infty} x\cdot p_a(x)$ and

(ii) the variance: $\sigma_x^2 \equiv \left\langle (x-\langle x\rangle)^2\right\rangle = \sum_{x=0}^{\infty}(x-\langle x\rangle)^2 p_a(x)$.