國立中央大學九十二學年度碩士班考試入學招生試題卷　　共一頁 第一頁

系所別：天文研究所　　科目：應用數學

(1) (25 points)

(a) (5 points) Describe the Frobenius' method (or series method) to solve an ordinary differential equation.

(b) (10 points) Find the solution of the hypergeometric equation near $t = 0$

$$t(t-a)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + (3t-a)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0$$

(c) (10 points) Find the second solution of the above equation.
[Hint: assume the second solution can be expressed as $x_2(t) = x_1(t)f(t)$, where $x_1(t)$ is the solution found in (b).]

(2) (25 points)
Solve the one dimensional inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \sum_{k=1}^{\infty} \frac{1}{k^2}\sin(2kt)\sin(kx)\,,$$

with the boundary conditions $u(t,0) = u(t,\pi) = 0$, and initial conditions $u(0,x) = \sin(2x)$ and $u_t(0,x) = 0$.
[Hint: expand u in a Fourier sine series of x, i.e., let $u(t,x) = \sum_{k=1}^{\infty} f_k(t)\sin(kx)$.]

(3) (20 points)

(a) (10 points) Mathematically, what is the meaning of a conservative vector field $\mathbf{F}$? Find out which of the following forces is/are conservative?
$\mathbf{F}_1 = -x\,\hat{e}_x - y\,\hat{e}_y - 2z\,\hat{e}_z$,
$\mathbf{F}_2 = -x/r^3\,\hat{e}_x - y/r^3\,\hat{e}_y + 2z/r^3\,\hat{e}_z$,
$\mathbf{F}_3 = -x/r^3\,\hat{e}_x - y/r^3\,\hat{e}_y - z/r^3\,\hat{e}_z$,
where $r^2 = x^2 + y^2 + z^2$.

(c) (10 points) Find the potential of the conservative force
$\mathbf{F} = (n-1)\cos\theta\sin\phi/r^n\,\hat{e}_r + \sin\theta\sin\phi/r^n\,\hat{e}_\theta - \cot\theta\cos\phi/r^n\,\hat{e}_\phi$,
where $r^2 = x^2 + y^2 + z^2$ and $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\phi$ are the unit vectors of the coordinate axes in spherical coordinates.

(4) (15 points)
Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & x & 0 \\ x & 2 & x \\ 0 & x & 1 \end{pmatrix}$$

and discuss the cases when $x = 0$ and $x = 1$.

(5) (15 points)
Evaluate the integral ($a^2 < 1$)

$$I = \int_0^{2\pi} \frac{\mathrm{d}\phi}{(1 + a\cos\phi)^2}\,.$$

[Hint: use Cauchy's integral formula for the nth derivative of a function, which states that

$$f^{(n)}(z) = \frac{1}{2\pi i}\oint_C \frac{n!\,f(z')\,\mathrm{d}z'}{(z'-z)^{n+1}}\,,$$

where z is inside the contour C and $f(z)$ is analytic inside C.]