參考用

1. If A and B are two events and the probability $P(B) \neq 1$, prove that

   (a) $P(A|\bar{B}) = \frac{P(A)-P(AB)}{1-P(B)}$, where $\bar{B}$ denotes the event complementary to B;

   (b) $P(AB) \geq P(A) + P(B) - 1$; and

   (c) $P(A) >$ or $< P(A|B)$ according as $P(A|\bar{B}) >$ or $< P(A)$. (15pts.)

2. A population consists of all the positive integers and the probability of obtaining the integer r from the population is

   $$P(r) = k(1-\theta)^{r-1}, (r = 1, 2, 3, \cdots), \text{ where } 0 < \theta < 1.$$

   (a) Determine the constant k and the <u>mean</u>, the <u>Variance</u> and <u>mode</u> of this population.

   (b) Show that if $\theta = 1 - (\frac{1}{2})^{1/n}$ for some positive integer n, then the <u>median</u> of the distribution may be consider to be $n + \frac{1}{2}$. (15 pts.)

3. Let $g(X)$ be a non-negative function of the random variable X and k is a positive constant.

   (a) Show that $P\{g(X) \geq k\} < \frac{1}{k}E\{g(X)\}$.

   (b) If $EX = \mu, VarX = \sigma^2$ and $E\{(X-\mu)^4\} = (1+\alpha^2)\sigma^4$, then, for any given constant c , prove that $P\left[1 - c - \lambda(\alpha^2 + c^2)^{\frac{1}{2}} \leq (\frac{X-\mu}{\sigma})^2 \leq 1 - c + \lambda(\alpha^2 + c^2)^{\frac{1}{2}}\right] \geq 1 - \frac{1}{\lambda^2}$ for any $\lambda > 0$. (15 pts.)

4. The joint distribution of the random variables X and Y is defined by a probability density function proportional to

   $$y^{\beta}(1-x)^{\alpha}, \text{ for } 0 \leq X \leq 1;\ 0 \leq Y \leq X,$$

   the parameters α and β being each > -1.

   (a) Find the marginal distributions of X and Y and evaluate their means and variances.

   (b) Determine the conditional distribution of X, given $Y = y$, and that of Y, given $X = x$. (15 pts.)

5. Let $\{X_n\}$ be a sequence of random variables. Suppose that X_k depends only on X_{k-1}, X_{k+1}, but that it is independent of all the other random variables $(k = 2, 3, \cdots)$. Show that if $V(X_i) \leq N < \infty$ $(i = 1, 2, \cdots)$, then

   $$\bar{X}_n - \bar{\mu}_n \longrightarrow 0 \text{ in probability,}$$

   $$\text{where } \bar{X}_n = \tfrac{1}{n}\sum_{i=1}^{n} X_i \text{ and } \bar{\mu}_n = \tfrac{1}{n}\sum_{i=1}^{n}\mu_i \text{ with } \mu_i = EX_i. \quad (15 \text{ pts.})$$

6. (a) State one version of Central Limit Theorem.

   (b) The round-off error to the second decimal place has the uniform distribution on the interval (-0.05, 0.05). Use (a) to find the probability, approximately, that the absolute error in the sum of 1,000 number is less than 1.82 . (15 pts.)

7. For fixed $t \in (0, 1)$, generate a sequence of independent random variables $\{U_1, \cdots, U_N\}$ uniformly distributed on (0, 1), where the random variable N is defined as follows:

   if $U_1 \geq t$, then $N = 1$;

   if not, then N is the first time such that U_N exceeds the previous variable, i.e. $U_N > U_{N-1}$ and $U_{N-1} < U_{N-2} < \cdots < U_1 < t$. Denote $g(t) = P(N \text{ is odd } |t)$, the probability that N is odd given $t \in (0, 1)$.

   (a) Show that $P\{N \text{ is even } |t\} = \int_0^t g(u_1)du_1$.

   (b) Find $g(t)$. (10 pts)