所別：數學系碩士班 甲組(一般生) 科目：線性代數
(學位在職生)

(7 problems in 1 page)

Problem 1. Let A be a complex $n\times n$ matrix. Let A^* be the *conjugate transpose* of A.

(a) (5%) Suppose that A is *self-adjoint* (that is, $A^* = A$). Prove that every eigenvalue of A is real.

(b) (5%) Suppose that A is *unitary* (that is, $A^*A = AA^* = I$). Prove that every eigenvalue λ of A has $\lambda\bar{\lambda} = 1$

Problem 2. (15%) Let A be a real 2×2 *symmetric* matrix. Prove that there exists a 2×2 real *orthogonal* matrix Q (that is, $Q^tQ=QQ^t=I$) such that

$$Q^tAQ = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \text{ where } \lambda_1 \text{ and } \lambda_2 \text{ are eigenvalues of } A.$$

Problem 3. The linear transformation $T: V \to V$ from a vector space V to itself satisfies $T^2 = T$ (that is a *projection*). Let I and O denote the identity and zero transformations on V. For each of the following give either a proof or a counterexample:

(a) (5%) If λ is an eigenvalue of T then $\lambda \in \{0,1\}$.

(b) (5%) $T=I$ or $T=O$.

Problem 4. Let W_1 and W_2 be subspaces of the finite dimensional vector space V over the field F. Prove that

(a) (5%) $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$ and $W_1 \cap W_2$ are subspaces of V.

(b) (10%) $\dim(W_1)+\dim(W_2)=\dim(W_1+W_2)+\dim(W_1\cap W_2)$.

Problem 5. (15%) Let V be a finite-dimensional vector space over the field F and let T be a *diagonalizable* linear transformation from V to itself. Let $c_1, c_2, \cdots, c_k$ be all the distinct eigenvalues of T. Prove that the minimal polynomial for T is the polynomial

$$p(x) = (x-c_1)(x-c_2)\cdots(x-c_k).$$

Problem 6. (10%) Prove that every matrix A such that $A^2 = A$ is *diagonalizable*.

Problem 7. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 6 & -4 & -4 \\ -6 & 6 & 6 \end{pmatrix}$ and $v = A^{2006}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(a) (15%) Find the eigenvalues and eigenvectors for the matrix A. Show your work.

(b) (10%) Calculate the vector v.