參考用

請作題 1 - 5, 及選作題 6, 7, 8 中任二題.

1. (20 points)

(a) Is the function $f(z) = \begin{cases} \frac{xy(x+iy)}{x^2+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$ differentiable at 0? Justify your answer.

(b) Show that $f(z) = x^2 + iy^2$ is differentiable at all points on the line $y = x$, but it is nowhere analytic.

2. Show that the series $\sum_{k=1}^{\infty} 1/(k^2 + z)$ converges and defines an analytic function on the right half-plane $\mathrm{Re} z > 0$. (10 points)

3. State Liouville's Theorem, the Maximum-Modulus Theorem, the Open Mapping Theorem, Schwarz' Lemma, and Morera's Theorem. (25 points)

4. Let $z_0$ be an isolated singularity of $f$. Prove

(a) $z_0$ is a removable singularity if and only if $\lim_{z \to z_0}(z - z_0)f(z) = 0$.

(b) $z_0$ is a pole of order $k \geq 1$ if and only if $\lim_{z \to z_0}(z - z_0)^k f(z) \neq 0$ and $\lim_{z \to z_0}(z - z_0)^{k+1} f(z) = 0$. (15 points)

5. Evaluate the integral $\int_0^{\infty} \frac{\cos x}{1+x^2} dx$. (10 points)

6. Show that if $f$ is analytic in a region $D$ and if $|f|$ is constant there, then $f$ is constant. (10 points)

7. If an entire function $f$ satisfies $|f(z)| \leq A|z|^c$ for some positive constants $A$ and $c$ and for all sufficiently large $z \in \mathbb{C}$, then $f$ is a polynomial of degree at most $n = [c]$. Prove it by showing that all the coefficients $C_k$, $k > n$, in the power series expansion of $f$ are 0. (10 points)

8. Show that $\int_{\Gamma_R} e^{iz^2} dz \to 0$ as $R \to \infty$ where $\Gamma_R$ is the circular segment: $z = Re^{i\theta}, 0 \leq \theta \leq \pi/4$. (10 points)