0, 2

所別：通訊工程學系碩士班 乙組 科目：工程數學

1. Let $T: R^3 \to R^3$ be a linear transformation defined by $T(a_1, a_2, a_3) = (a_1 - 2a_3, a_1 + 4a_2, 2a_2 + a_3)$.
   (a) Prove that T is a linear transformation. (5%)
   (b) Find bases for both the null space (kernel) of T and the range of T. (5%)
   (c) Determine whether T is one-to-one. (5%)

2. Let $M_{m\times n}$ represent the set of all $m \times n$ matrices. For $A \in M_{m\times n}$, let A_i denote the i-th column of A where $i \in \{1, 2, \cdots, n\}$. Is the set of all $m \times n$ matrices having $\sum_{i=1}^{n} A_i = 0_{m\times 1}$ a subspace of $M_{m\times n}$? If the answer is "yes", prove it and find a basis for this subspace. If the the answer is "no", justify your answer. (10%)

3. Suppose that $v_1, v_2, \cdots, v_n$ are pairwise orthogonal vectors (that is, $< v_i, v_j >= 0 \ \forall \ i \neq j$) in an inner product space. Prove that $\|v_1 + v_2 + \cdots + v_n\|^2 = \|v_1\|^2 + \|v_2\|^2 + \cdots + \|v_n\|^2$. (10%)

4. Find A^{100}, if
   (a) $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix}$. (10%)
   (a) $A^{50} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix}$. (5%)

5. Two random variables X and Y are independent and uniformly distributed on [0, 1]. Another two random variables W and V are given by

$$W = \sqrt{-2\log X}\,\sin(2\pi Y)$$

and

$$V = \sqrt{-2\log X}\,\cos(2\pi Y).$$

Show that W and V are independent and each has the standard normal distribution. (10%)

6. Define the characteristic function of a random variable X as

$$\Phi_X(\omega) = E[e^{j\omega X}],$$

where $j = \sqrt{-1}$.

(a) Let X and Y be two independent, identically, distributed (i.i.d.) random variables, and $Z = X + Y$. Show that

$$\Phi_Z(\omega) = \Phi_X(\omega) \cdot \Phi_Y(\omega). \quad (10\%)$$

(b) If X and Y are uniformly distributed on [-1/2, 1/2]. Find the characteristic function of Z and the PDF of Z. (10%)

注意：背面有試題

7. In a noisy ternary communication channel, three possible symbols {-1, 0, +1} are transmitted to the receiver and the receiver decides the results as {-1, 0, +1}. A "-1" is sent three times more frequently than a "0", and a "0" is sent two times more frequently than a "+1". The probability of deciding "-1" when transmitting "-1" (we say "correct transmission" for transmitting "-1".) is $1-\alpha$ and the probability of deciding "0" when transmitting "-1" is α. The probabilities of deciding "-1" or "+1" when transmitting "0" both are β and the probability of deciding "0" when transmitting "0" is $1-2\beta$. The case for transmitting "+1" is the same as that for transmitting "-1".
   (a) Find the probability of transmitting "0" given that "0" is decided. (5%)
   (b) Find the average probability of correct transmission in this channel. (5%)

8. Consider the recursions composed of random variables $X(n)$ and $Z(n)$ for $n = \ldots,-1,0,1,\ldots$. Assume that $E[Z(n)]=0$, $E[Z^2(n)]=\sigma^2$, $E[Z(n)Z(j)]=0$ for all $n \neq j$, and $E[Z(n)X(n-k)]=0$ for $k=1,2,\ldots$.
   (a) If the recursion is given by
   $$X(n)=\alpha X(n-1)+\beta_0 Z(n),$$
   find $R_X(k)=E[X(n)X(n-k)]$ for $k=0,1,2,\ldots$. (5%)
   (b) If the recursion is given by
   $$X(n)=\beta_0 Z(n)+\beta_1 Z(n-1),$$
   find $R_X(k)=E[X(n)X(n-k)]$ for $k=0,1,2,\ldots$. (5%)