國立中央大學八十八學年度轉學生入學試題卷

數學系　三年級　　科目：高等微積分　　共 1 頁　第 1 頁

$\mathbb{R}$ denotes the real number system.

1. (15%) Let

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0, \end{cases}$$

and let $G \subset \mathbb{R}^2$ be the graph of f, that is,

$$G = \{(x, f(x)) : x \in [0,1]\}.$$

Is G a compact subset of $\mathbb{R}^2$? Prove your answer.

2. (20%) Let

$$f_n(x) = e^{-nx} \sin n\pi x, \qquad x \in [0,1].$$

Show that $\lim_{n\to\infty} f_n(x) = 0$ pointwise, but not uniformly. Is

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0?$$

3. (15%) Let

$$f(x) = \int_0^{x^2} e^{t^3}\,dt.$$

Prove that

$$3\int_0^1 e^{x^3}\,dx - 12\int_0^1 x^3 f(x)\,dx = e - 1.$$

4. (a) (5%) State the Cauchy criterion for an infinite series to be convergent.

   (b) (15%) Let $a_n > 0$ for each n, $S_n = \sum_{k=1}^{n} a_k$ and assume that $\sum_{n=1}^{\infty} a_n$ is divergent.

   Prove that the series $\sum_{n=1}^{\infty} \dfrac{a_n}{S_n}$ is also divergent.

5. Let

$$f(x,y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) (10%) Prove that f_x exists at every point $(x_0, y_0) \in \mathbb{R}^3$.

   (b) (5%) Is f_x continuous at $(0,0)$? Prove it.

   (c) (5%) Let $a = (x_0, y_0) \in \mathbb{R}^2$. Give the definition of the statement "f is differentiable at a".

   (d) (10%) Prove that f is not differentiable at $a = (0,0)$.

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