Linear Algebra

1. (15%) Let T be a linear transformation from R^n to R^n. Let v be a vector in R^n with $v \neq 0$. Show that there exists a polynomial $p(x)$ such that $p(T)v = 0$.

2. (15%) Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ be a 4×4 matrix over the complex numbers. Find all eigenvalues of A.

3. (15%) Let λ and μ be two distinct eigenvalues of a symmetric matrix A. Let u and v be eigenvectors associated with λ and μ respectively. Show that u and v are orthogonal.

4. (15%) Let T be a linear transformation from a vector space V to a vector space W. Let P and Q be subspaces of V and W respectively. Show that $T(P)$ and $T^{-1}(Q)$ are subspaces of W and V respectively.

5. (15%) Let
$S = \{(x_1, x_2, x_3, x_4, x_5) \in R^5 : x_1 + x_2 + x_3 + x_4 + x_5 = 0, \quad x_1 - x_2 - x_3 + x_4 - x_5 = 0\}$.
Find an orthonormal basis for the space S.

6. (15%) Find the inverses of the following matrices if it is possible.

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 3 & 2 \end{bmatrix}.$$

7. (10%) Let $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ -1 & -2 & 2 & 1 & 2 & 3 \\ 0 & 0 & 5 & 2 & 4 & 6 \\ 2 & 4 & 6 & 2 & 4 & 6 \\ 1 & 1 & 1 & 2 & 2 & 6 \end{bmatrix}$. Find the rank of A.