國立中央大學103學年度碩士班考試入學試題卷

類別：資工類　　科目：離散數學與線性代數　　共 3 頁　第 1 頁　　＊請在試卷答案卷（卡）內作答

本科考試禁用計算器

複選題(答對給5分，答錯或不答給0分，不倒扣)

1.(5%) If square matrix $A = [a_{ij}]$ has eigenvalues $\lambda_1, \lambda_2, .., \lambda_n$ and corresponding linearly-independent eigenvectors $e_1, e_2, .., e_m$. Which are correct ? (A) If $m = n$, A is diagonalizable. (B) If A is diagonalizable, it is possible for $\lambda_i = \lambda_j$, $i \neq j$. (C) $a_{11} + .. + a_{nn} = \lambda_1 + ... + \lambda_n$. (D) $\{e_1, e_2, .., e_m\}$ is always a basis of a subspace of R^n. (E) If $e_i = a+ib$, then a and b are always linearly independent.

2.(5%) If A is a $m \times n$ matrix, W is the set of all columns of A, and $W^{\perp}$ is the orthogonal complement of W, then (A) W is always a subspace. (B) $W^{\perp}$ is always a subspace. (C) $(W^{\perp})^{\perp} = W$. (D) The intersection of W and $W^{\perp}$ is always not an empty set. (E) dim $W^{\perp}$ + dim $(W^{\perp})^{\perp} = n$.

3.(5%) If inconsistent linear system $Ax = b$ has a least-square solution $\hat{x}$ and $A = [a_1\ a_2\ ..\ a_n]$, which are correct ? (A) $\hat{x}$ is always existed. (B) $\hat{x}$ is always unique. (C) $A\hat{x}$ is always in $span\{a_1, a_2, .., a_n\}$. (D) $\hat{x} = (A^TA)^{-1}A^Tb$. (E) $A\hat{x} - b$ is orthogonal to rows of A.

4.(5%) If A can be QR factorization, which are correct? (A) A is a square matrix. (B) A has linearly independent columns. (C) $QQ^T = I$, where I is an identity matrix. (D) Q has positive entries on its diagonal. (E) R is invertible.

5.(5%) If A is a symmetric matrix. Which are correct? (A) A is always diagonalizable. (B) A has linearly independent eigenvectors. (C) A always has real eigenvalues. (D) The eigenvalues of A are always positive. (E) A can always be spectral decomposed, $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + .. + \lambda_n u_n u_n^T$.

6.(5%) Suppose that $n \times n$ matrix A is invertible. Which of the following statements are true ? (A) The row vectors of A should be linearly independent. (B) det(A)=0 (C) A does not have eigenvalue 0. (D) The rank of A is n. (E) Ax=0 has nontrivial solution.

7.(5%) Suppose that A and B are two square matrices. Determine which of the following are true. (A) det(AB)=det(A)det(B). (B) det(A)=det(A^T). (C) det(AB)=det(BA). (D) det(A^{-1})=1/det(A) if A^{-1} exists. (E) det(A^k)=(det(A))^k.

8.(5%) Determine which of the following statements are true. (A) $W=\{\{x,y\} \mid x+y=1\}$ is a subspace. (B) $W=\{\{x,y\} \mid x>3y\}$ is a subspace. (C) Suppose that W_1 and W_2 are two subspaces. Then $W_1 \cap W_2$ is also a subspace. (D) Suppose that W_1 and W_2 are two subspaces. Then $W_1 \cup W_2$ is also a subspace. (E) $W=\{\{a_0+a_1x+a_2x^2\} \mid a_0, a_1, a_2$ are scalar, and $a_2 \neq 0\}$ is a subspace.

9.(5%) Suppose that $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are the standard matrices of the two transformations, T_1 and T_2, respectively. Which of the following statements are correct? (A) T_1 is a linear transformation. (B) T_1 is the transformation that counter-clockwisely rotates each vector through an angle θ. (C) T_1 is the transformation that clockwisely rotates each vector through an angle θ. (D) $A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. (E) T_2 is the transformation that orthogonally projects each vector onto x- axis.

10. (5%) Suppose that two $n \times n$ matrices, A and B, are orthogonal. Which of the following statements are correct? (A) A^{-1} is orthogonal. (B) det(A) = 1 or −1. (C) Columns of A form an orthonormal set in R^n with the Euclidean inner product. (D) AB is an orthogonal matrix. (E) $A^TABB^T=I$, where I is an identity matrix.

多選題(每一選項單獨計分，答錯每選項倒扣1分)

11. (5%)Given a function f from A to B and $f(a) = b$ (where $a \in A$ and $b \in B$, which of the following statements are correct?

(A) A is the domain of f.

(B) B is the range of f.

(C) $f \in B^A$.

(D) b is the image of a under f.

(E) a is the pre-image of b under f.

12. (5%)Given the following piece of code, which of the following statements are correct?

```
int Fibonacci(int n)
begin
   if (n == 0) or (n == 1)
      return 1;
   else
      return Fibonacci(n - 1) + Fibonacci(n - 2);
   endif
end
```

(A) This function computes Fibonacci series.

(B) This is a recursive function.

參考用

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(C) The growth rate of this function is proportional to the current value of the function.
(D) The time complexity $T(n)$, where n is the argument of the function, is also a Fibonacci series.
(E) The time complexity can be reduced if the function is designed to be iterative.

13. (5%)Which of the following statements about basic number theory are correct?
(A) $a|0$ for any a.
(B) If a and b are positive integers, then there exist integers s and t such that $\gcd(a, b) = sa + tb$.
(C) If m is a prime integer, then an inverse of a modulo m exists.
(D) To satisfy equations $x \equiv 1 \pmod 3$, $x \equiv 2 \pmod 5$, and $x \equiv 3 \pmod 7$, 52 is the only solution.
(E) If p is prime and a is an integer not divisible by p, then $a^p \equiv a \pmod p$.

14. (5%)There are 80 students in the class, which of the following statements are correct according to the pigeonhole principle?
(A) There is at least one student's birthday in each and every week.
(B) There are at most 2 students whose birthday are in the same week.
(C) There is at least one week in which at least 2 students have birthday.
(D) There are at least seven students who were born with the same astrological sign (星座).
(E) There is at most one astrological sign (星座) to which more than seven students belong.

15. (5%)Which of the following statements about hypercube are correct?
(A) Q_0 has 1 node.
(B) For any integer n > 1, the hypercube Q_n is a simple graph consisting of four copies of Q_{n-1} connected together at corresponding nodes.
(C) Q_n has 2^n nodes.
(D) The recurrence relation of the number of edges for Q_n, denoted as E(n), is $E(n) = 2E(n - 1) + 2^n$.
(E) Q_n has $n \cdot 2^{n-1}$ edges.

16. (5%)The worst case time complexity for Euclid's algorithm to find $\gcd(a,b)$ (a,b in Z) is (suppose $n=(max(a,b))$): (A) $\theta(n)$. (B) $O(\log(n))$. (C) $\theta(\log(n))$. (D) $\theta(n \log(n))$. (E) none of the above.

17. (5%)Among the following options, which are necessary but not sufficient conditions for the corresponding goals?
(A) "A set of total-order predicates" for "applying mathematical induction proof on those predicates".
(B) Assume g, f are functions mapping from A to B domains, and C to D, respectively. "B,C are the same domain" for "$(f \circ g)$ composition is possible".
(C) "existing exponential time algorithm" for "intractable (NP) problems".
(D) "X is a student and X is in the class" for "X is in the class only if X is a student".
(E) "f is $O(g)$" for "f is $\theta(g)$".

18. (5%)To analyze the complexity of the following procedure P, We will use the following assumptions: Suppose P and Q are both procedures. Q take $\theta(\sqrt{m})$ time to compute, where m is the size of input; each statement line in and outside the loop counts 1 step.

Procedure P(array1[$a_1, a_2, \ldots, a_n$])
1. if n<5 exit.
2. declare initially new empty array2, array3;
3. call Q (array1);
4. for (i=1 to n) {
5. if ((i mod 5) =1) { insert a_i into array2;}
6. if ((i mod 5) =3) { insert a_i into array3;}
}
7. call P(array2);
8. call P(array3);
9. return();

Suppose n is a number of power of 5, What can be the time complexity level of the procedure P in the question above?

(A) $\theta(n)$ (B) $\theta(\sqrt{n}\log n)$ (C) $\theta(\sqrt{n})$ (D) $O(n^{\log_5 2})$

(E) $O(\sqrt{n}\log n)$

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19. (5%)We want to count the number of ways to climb n stairs if the climbing person can take one stair or two stairs at a time. What of the following can be the recurrence relation for our question? (initial condition: $a_0 = 1; a_1 = 1;$)

(A) $a_n = a_{n-1} + 1$.

(B) $a_n = a_{n-2} + 3$.

(C) $a_n = 2a_{n-2} + 1$.

(D) $a_n = a_{n-1} + a_{n-2}$.

(E) none of the above.

20. (5%)To solve the recurrence relation in 19., what can be the generating function $f(z)$?

(A) $f(z) = 1/(1 - z - z^2)$.

(B) $f(z) = z/(1 - z - z^2)$.

(C) $f(z) = \frac{1}{\sqrt{5}} \cdot \left(\frac{1}{1-((1+\sqrt{5})/2)z} - \frac{1}{1+((1-\sqrt{5})/2)z}\right)$

(D) $f(z) = \frac{2}{1-z} + \frac{2-3z}{(1+2z)^2}$.

(E) none of the above.