國立中央大學105學年度碩士班考試入學試題

所別： 資工類

科目： 離散數學與線性代數

本科考試禁用計算器 *請在答案卷(卡)內作答

離散數學(50%)

多選題（每一小題答對給 1 分、答錯扣 1 分、不答 0 分）

1. What is the value of $\left\lceil \left(\frac{1}{2}\right)^{\lfloor 5/2 \rfloor} \right\rceil$?

   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) 0.5

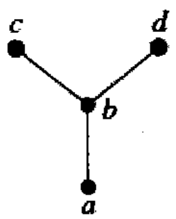
2. Let f and g be the function from the set of integers to itself, defined by $f(x) = 2x + 1$ and $g(x) = 3x + 4$. What is the composition of f and g (i.e., $f \circ g$)?.

   (a) $6x + 9$
   (b) $6x + 8$
   (c) $6x + 7$
   (d) $6x + 6$
   (e) $6x + 5$

3. Compute $3^{302} \bmod 5$

   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) 4

4. The following Hasse diagram represents a poset.



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Which of the following matrices best represents the relation of this poset?

(a) $\begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}$ (b) $\begin{bmatrix} 1&1&1&1 \\ 0&1&1&1 \\ 0&0&1&1 \\ 0&0&0&1 \end{bmatrix}$ (c) $\begin{bmatrix} 0&1&1&1 \\ 0&0&1&1 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}$ (d) $\begin{bmatrix} 1&1&1&1 \\ 0&1&1&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}$ (e) $\begin{bmatrix} 1&1&1&1 \\ 0&1&1&1 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}$

5. Which of the following pairs between the graph and the number of its vertices is incorrect?
   (a) K_n $n(n-1)/2$
   (b) C_n n
   (c) W_n $n+1$
   (d) Q_n $n2^{n-1}$
   (e) $K_{m,n}$ mn

6. Among the following options, which are necessary but not sufficient conditions for the corresponding goals?
   (a) "Existing an equivalence relation on a set S" for "can form a partition on S".
   (b) Between 2 graphs G, H, both with n nodes: "Existing same number of length-1 ~ length-(n/2) paths" for "G is isomorphic to H".
   (c) "Existing an 1-to-1 mapping from A to B" for "A and B have the same cardinality".
   (d) "P is true" for "P→Q is true".
   (e) "R is a partial-order set" for "R is a well-ordered set".

7. To analyze the complexity of the following procedure P, We will use the following assumptions: Suppose P and Q are both procedures. Q will take $\theta(\sqrt[3]{m})$ time to partition a length-m array into length-$\lfloor m/8 \rfloor$ arrays, $B_1, B_2, \ldots, B_8$, where m is the size of input; each statement line in and outside the loop counts 1 step.

Procedure $P(\mathrm{A}[a_1, a_2, \ldots, a_n])$
Declare $B_1, B_2, \ldots B_8$ as initially empty arrays;
1. if n<8 exit.
2. call Q (A); /* and get $B_1, B_2, \ldots$ */
3. call $P(B_1)$;
4. call $P(B_8)$;
5. return

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Suppose n is a number of power of 8, What of the following options are true about the number of steps ($p(n)$) and complexity (C_p) of the procedure P in the question above?
(A, B are constants)

(a) $p(n) = 2p(n/8) + An^{1/3} + B$

(b) $p(n) = 3p(n/8) + An^{1/2} + B$

(c) $C_p = \theta(n \log n)$

(d) $C_p = \theta(\sqrt[3]{n} \log n)$

(e) $C_p = \theta(\sqrt[3]{n})$

8. Suppose every element in A is different from others in A, and $|A| = 7$, $a \in A$. What about the following set C is true?

$C = \{B \mid (((a \notin B) \rightarrow (|B| = 2)) \vee ((a \in B) \rightarrow (|B| = 3))) \wedge (B \subseteq A)\}$

(a) $|C| = 2\binom{6}{2}$

(b) $|C| = \binom{6}{2} + \binom{7}{2}$

(c) $|C| = \binom{6}{2} + \binom{7}{3}$

(d) $C \supseteq A$

(e) $|C| > |A|$

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9. The predicate $G(x,y)$ means student x and y have the same grade. What can be the correct English statements for the logic expression –

"$\exists x, y, \forall z, (x \neq y) \wedge G(x,y) \wedge (G(x,z) \rightarrow ((x = y) \vee (x = z)))$."

(a) Some student has the same grade with exactly one other student.

(b) There are two students who have a certain same grade, and all other students do not have that same grade.

(c) All students have either one or the other possible grades.

(d) Every student can find exactly other one student that has the same grade.

(e) No student can have the same grade as more than one other student.

10. What options are true when using generating function to solve the recurrence relation: $a_n = 2a_{n-1} - a_{n-2} + 2^{n-2}, a_0 = 1, a_1 = 2$. ($g(z)$ is the generating function)

(a) $g(z)=1/(1-2z)$

(b) $g(z)=(1-2z+z^2)/(1-2z)$

(c) $g(z)=\frac{1}{\sqrt{5}}.(\frac{1}{1-((1+\sqrt{5})/2)z}-\frac{1}{1+((1-\sqrt{5})/2)z})$

(d) $a_n=2^n$

(e) None of the above.

線性代數 (50%)

多重選擇題 (每一小題答對給1分、答錯扣1分、不答0分)

11. Define sum $\mathbf{u}+\mathbf{v}=(u_1+v_1, u_2+v_2, \ldots.u_n+v_n)$ and scalar multiple $k\mathbf{u}=(ku_1, ku_2, \ldots, ku_n)$, where vectors $\mathbf{u}=(u_1, u_2, \ldots, u_n)$, $\mathbf{v}=(v_1, v_2, \ldots, v_n)$, and k is a scalar. Which of the following are subspaces of $\boldsymbol{R}^3$?
    (a) All vectors of the form $(a, 0, 0)$.
    (b) All vectors of the form (a, b, c) where $b=a+c$.
    (c) All vectors of the form (a, b, c) where $b=a\times c$
    (d) All vectors of the form $(a, 2a, 3a)$.
    (e) All vectors of the form $(a, 1, 3)$.

12. Suppose that A is a 5×6 matrix. Determine which of the following statements are true?
    (a) The rank of A is at most 5.
    (b) The rank of A^T is at most 5
    (c) The number of parameters in the general solution of $A\mathbf{x}=0$ is at most 6.
    (d) The nullity of A^T is at most 6.
    (e) None of the above

13. A linear transformation $T:\boldsymbol{R}^2\to\boldsymbol{R}^2$ such that

$$T(1, 1)=(1, 0, 1) \text{ and } T(2, 3)=(1, -1, 2).$$

Indicate which of the following statements are true?

(a) $T(14, 19)=(9, -5, 14)$

(b) $T(3, 4)=(2, -1, 3)$

(c) $T(4, 6)=(2, -2, 4)$

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(d) $T(0,1) = (0, -2, 6)$

(e) None of the above

14. Let $[T_1]=\begin{bmatrix}1 & -2 & 2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$, $[T_2]=\begin{bmatrix}1 & -3 & 4\\ -1 & 1 & 1\\ 1 & -2 & 5\end{bmatrix}$, $[T_3]=\begin{bmatrix}1 & 4 & -3\\ 2 & 7 & 1\\ 1 & 1 & 3\end{bmatrix}$, and

$[T_4]=\begin{bmatrix}1 & 4 & -3\\ 2 & 7 & 1\\ 0 & 1 & 3\end{bmatrix}$. Which linear operators are one-to-one?

(a) T_1 (b) T_2 (c) T_3 (d) T_4 (e) None of the above.

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15. Let $A=\begin{bmatrix}-\sqrt{3}/2 & 1/2\\ 0 & 0\end{bmatrix}$, $B=\begin{bmatrix}\sqrt{2} & -\sqrt{2}\\ \sqrt{2} & \sqrt{2}\end{bmatrix}$, and $C=\begin{bmatrix}3 & 0\\ 0 & -3\end{bmatrix}$. Which of the following are true?

(a) Matrix A is the standard matrix for the composition of linear operators stated below: a counterclockwise rotation of 60°, followed by an orthogonal projection on the x-axis.

(b) Matrix B is the stand matrix for the composition of linear operators stated below: a dilation with factor k=2, followed by a counterclockwise rotation of 45°.

(c) Matrix C is the standard matrix for the linear operator defined below: a reflection about the x-axis, followed by a contraction with factor k=1/3.

(d) Matrix C is the standard matrix for the linear operator defined below: a reflection about the x-axis, followed by a dilation with factor k=3.

(e) None of the above.

16. If $n\times n$ matrix A has r distinct eigenvalues $\lambda_1, .., \lambda_r$, $r < n$, then

(a) A is not diagonalizable.

(b) A has at most n eigenvectors.

(c) A is not invertible.

(d) A has at least r linearly independent eigenvectors.

(e) The sum of two eigenvectors may also be an eigenvector of A.

國立中央大學105學年度碩士班考試入學試題

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17. Let $\boldsymbol{A}$ be an $m \times n$ matrix with orthonormal columns. If $\boldsymbol{W}$ is a subset of $\boldsymbol{R}^n$ and $\boldsymbol{W}^{\perp}$ is the orthogonal complement of $\boldsymbol{W}$, then

    (a) If w is in **Nul**A (i.e., null space of A) and $\boldsymbol{v} \cdot \boldsymbol{w} = 0$, then $\boldsymbol{v}$ is in **ColA** (i.e., column space of A).
    (b) $\boldsymbol{W}^{\perp}$ is always a subspace.
    (c) $(\boldsymbol{W}^{\perp})^{\perp} = \boldsymbol{W}$.
    (d) If $\boldsymbol{v}$ and $\boldsymbol{w}$ is orthogonal, then $\boldsymbol{A}\boldsymbol{v}$ and $\boldsymbol{A}\boldsymbol{w}$ is orthogonal.
    (e) $m \geq n$.

18. If A is a $n \times n$ matrix with real entries.
    (a) A may have zero eigenvalues and eigenvectors.
    (b) A has only real eigenvalues if A is symmetric.
    (c) A can always be diagonalizable if A is symmetric.
    (d) A can always be spectral decomposed if A is symmetric.
    (e) A^TA may have complex eigenvalues $a+ib$ and $a-ib$.

19. If $\boldsymbol{Ax} = \boldsymbol{b}$ is an inconsistent system and $\boldsymbol{A}$ is a $m \times n$ matrix.
    (a) A has only r linearly independent columns, $r < n$.
    (b) The least-square solution of $\boldsymbol{A}$ is $(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b}$.
    (c) A may have no least-square solution.
    (d) If $\hat{\boldsymbol{u}}$ is a least-square solution of $\boldsymbol{A}$, then $(\boldsymbol{b} - \boldsymbol{A}\hat{\boldsymbol{u}})$ is orthogonal to every row of A.
    (e) If A can be QR factorized, then the least-square solution can be calculated as $\boldsymbol{R}^{-1}\boldsymbol{Q}\,\boldsymbol{b}$.

20. Find a singular value decomposition $\boldsymbol{A} = \boldsymbol{U\Sigma V}^{\mathrm{T}}$ with $\boldsymbol{U}$ and $\boldsymbol{V}$ being both orthogonal matrices, where $\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$. Which values are **not** in $\boldsymbol{U}$ or $\boldsymbol{V}$ matrices?

    (a) $1/\sqrt{2}$ (b) $1/\sqrt{3}$. (c) 1/2. (d) 1/3. (e) $1/(3\sqrt{2})$.

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