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科目： 線性代數

本科考試禁用計算器

Instructions: Show your work. The notations $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the rational number, real number and complex number fields, respectively.

1. Let

$$A = \begin{pmatrix} 2 & -1 & 3 & 1 & 4 \\ 4 & 2 & 1 & 3 & 11 \\ -6 & -1 & 2 & -11 & -14 \\ 8 & 4 & 8 & -8 & 26 \\ -2 & -7 & -11 & 53 & -17 \end{pmatrix}$$

denote a 5×5 matrix over $\mathbb{Q}$.

(a) (15%) Find a 5×5 lower triangular matrix L with diagonal entries all 1 and a 5×5 upper triangular matrix U such that

$$A = L \cdot U.$$

(b) (5%) Solve

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -25 \\ -20 \\ 12 \\ -90 \\ 138 \end{pmatrix}$$

for x_1, x_2, x_3, x_4, x_5.

2. (15%) Let n denote a non-negative integer. Let $c_0, c_1, \ldots, c_n$ denote $n+1$ mutually distinct scalars taken from $\mathbb{Q}$. Show that given any $d_0, d_1, \ldots, d_n \in \mathbb{Q}$, there exists a unique polynomial $f(x)$ of degree $\leq n$ with coefficients in $\mathbb{Q}$ such that

$$f(c_i) = d_i \qquad \text{for all } i = 0, 1, \ldots, n.$$

3. (15%) Given an infinite sequence $a_1, a_2, a_3, \ldots$ in $\mathbb{R}$, we simply write the sequence by $\{a_n\}_{n\geq 1}$. Let $\mathbb{R}^\infty$ denote the set of all sequences $\{a_n\}_{n\geq 1}$ in $\mathbb{R}$. Note that $\mathbb{R}^\infty$ is a vector space over $\mathbb{R}$ with vector addition $+$ and scalar multiplication $\cdot$ defined by

$$\{a_n\}_{n\geq 1} + \{b_n\}_{n\geq 1} = \{a_n + b_n\}_{n\geq 1} \qquad \text{for all } \{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1} \in \mathbb{R}^\infty,$$
$$\lambda \cdot \{a_n\}_{n\geq 1} = \{\lambda \cdot a_n\}_{n\geq 1} \qquad \text{for all } \lambda \in \mathbb{R} \text{ and } \{a_n\}_{n\geq 1} \in \mathbb{R}^\infty.$$

Show that $\mathbb{R}^\infty$ is an infinite-dimensional vector space over $\mathbb{R}$.

4. (15%) Let

$$A = \begin{pmatrix} -3 & -2 & 0 & 0 \\ -2 & 1 & 8 & 0 \\ 0 & 2 & -3 & 6 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

denote a 4×4 matrix over $\mathbb{Q}$. Determine if A is diagonalizable and give a proof for your answer.

5. (15%) Let V denote a finite-dimensional vector space over a field $\mathbb{F}$. Assume that $T : V \to V$ is a diagonalizable linear operator. Prove that if W is a T-invariant subspace of V, then the linear operator $T|_W : W \to W$ given by

$$T|_W(w) = T(w) \qquad \text{for all } w \in W$$

is diagonalizable.

注意：背面有試題

國立中央大學108學年度碩士班考試入學試題

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6. (20%) Let V denote a vector space over the complex number field $\mathbb{C}$ endowed with an inner product $\langle\ ,\ \rangle$. Recall that the norm of a vector $v \in V$ is defined as

$$||v|| = \sqrt{\langle v, v\rangle}.$$

Let W denote a finite-dimensional subspace of V. Prove that there exists a unique linear operator $P : V \to W$ such that

$$||v - P(v)|| \leq ||v - w|| \qquad \text{for all } v \in V \text{ and } w \in W.$$

注意：背面有試題