

# 國立中央大學 109 學年度碩士班考試入學試題

所別： 數學系 碩士班 數學組(一般生)

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科目： 高等微積分

本科考試禁用計算器

\*請在答案卷(卡)內作答

Show your work for all of the questions below. Your method must justify your answer. No credit for answers without supporting work.

**Proofs:** (In all of the questions below,  $[a, b]$  denotes a bounded closed interval in  $\mathbb{R}$  with  $a < b$ .)

1. (10%) Prove that  $x \cos \frac{1}{x}$  is uniformly continuous on  $(0, \infty)$ .
2. (15%) Let  $f : [a, b] \rightarrow \mathbb{R}$  be an increasing function. Show that  $f$  is (Riemann) integrable on  $[a, b]$ .
3. (15%) Suppose that  $f$  is continuous on  $[a, b] \subset \mathbb{R}$ ,  $g$  is integrable on  $[a, b]$  and  $g(x) \geq 0$  for all  $x \in [a, b]$ . Prove that

$$\int_a^b f(x)g(x) dx = f(c) \int_a^b g(x) dx \quad \text{for some } c \in [a, b].$$

4. (15%) Suppose that  $E$  is a nonempty subset of  $\mathbb{R}$  and that  $f_n \rightarrow f$  uniformly on  $E$  as  $n \rightarrow \infty$ . Prove that if each  $f_n$  is continuous on  $E$ , then  $f$  is continuous on  $E$ .
5. (15%) Prove that  $\sum_{n=1}^{\infty} \sin(x/n^2)$  converges uniformly on any bounded interval of  $\mathbb{R}$ .
6. (15%) Let  $C[a, b]$  denote the space of continuous functions  $f : [a, b] \rightarrow \mathbb{R}$ . Define

$$\|f\| := \sup_{x \in [a, b]} |f(x)|.$$

Show that  $\rho(f, g) := \|f - g\|$  is a metric on  $C[a, b]$ , and the metric space  $(C[a, b], \rho)$  is complete.

7. (15%) Prove that

$$f(x, y) = \begin{cases} \frac{3x^3 - 2x^2y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is continuous and has first-order partial derivatives everywhere on  $\mathbb{R}^2$ , but  $f$  is not differentiable at  $(0, 0)$ .

參考用