國立中央大學110學年度碩士班考試入學試題

所別： 資工類　　　　　　　　　　　　　　　　　　　　　　　　　　　　共7頁　第1頁

科目： 離散數學與線性代數

本科考試禁用計算器　　　　　　　　　　　　　　　　　　　　　　　　　　＊請在答案卷(卡)內作答

離散數學多選題（50 分）－第一題到第十題為，每題五分（ABCDE 每一選項單獨計分）答對每一選項的分數為 1 分，答錯每一選項倒扣 1 分，倒扣至離散數學多選題整大題 0 分為止。

1. Which of the following logic equivalence statements are incorrect?
   A. $\forall x P(x) \wedge \forall x Q(x) \equiv \forall x(P(x) \wedge Q(x))$.
   B. $\forall x P(x) \vee \forall x Q(x) \equiv \forall x(P(x) \vee Q(x))$.
   C. $\exists x P(x) \wedge \exists x Q(x) \equiv \exists x(P(x) \wedge Q(x))$.
   D. $\exists x P(x) \vee \exists x Q(x) \equiv \exists x\big(P(x) \vee Q(x)\big)$.
   E. $\forall x(P(x) \rightarrow A) \equiv \forall x P(x) \rightarrow A$. (Assume that x does not occur as a free variable in A and the domain of x is nonempty.)

2. 小英、江哥、柯 P 和 3Q 哥在聊離散數學，以下哪幾句敘述有誤？
   A. 小英對江哥說：從我的黨員到你的黨員間存在有 injective function 的映射，所以我的黨比你的黨大。
   B. 柯 P 對小英說：從我的黨員到你的黨員間存在有 bijective function 的映射，所以我的黨比你的黨大。
   C. 江哥說：從自然數集N到我的黨員間存在有 surjective function 的映射，所以我的黨比小英跟柯 P 兩個黨加起來還大。
   D. 3Q 哥對江哥說：從有理數集Q到我的黨員間存在有 surjective function 的映射，所以我的黨才是最大的。
   E. 以上皆是。

3. Which of the following statements about integers are incorrect?
   A. If $\gcd(a,b)=1$, then for any non-zero integer n, there is a pair of integers p and q such that $pa+qb=n$.
   B. The inverse of p module q exists only when $\gcd(p,q)=1$ and $q>1$.
   C. The system $\begin{cases} x \equiv a_1 (\text{mod } m_1) \\ x \equiv a_2 (\text{mod } m_2) \\ \vdots \\ x \equiv a_n (\text{mod } m_n) \end{cases}$ has a unique solution modulo m, where m_i are primes and $m = m_1 m_2 \cdots m_n$.
   D. If p is prime, then for every integer a we have $a^{p-1} \equiv 1 (\text{mod } p)$.
   E. In RSA, if Alice wants to send a secret message to Bob, Alice uses Bob's private key to encrypt the message and then send the ciphertext message to Bob. After Bob receives the ciphertext, Bob can use his public key to decrypt the ciphertext.

注意:背面有試題

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4. Figure 1 shows the Hasse diagram of a binary relation R. Which of the following statements are incorrect?

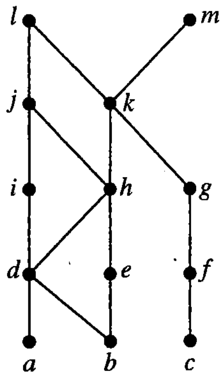

Figure 1: Hasse diagram of R.

A. cRh.
B. mRm.
C. dRa.
D. a is a minimal element.
E. R is a lattice.

5. Given the directed graph in Figure 2. Which of the following statements are correct?

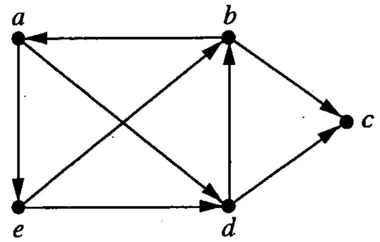

Figure 2. Directed graph G.

A. G is strongly connected.
B. G is weakly connected.
C. G has a Euler path.
D. G has a Hamilton path.
E. G is a planar graph.

6. Considering relation **R** defined as : $\{(x,y) \in N \times N \mid x$ does not have more prime factors than $y\}$, Which of the following statements are true?
A. **R** is partial ordering.
B. **R** is total-ordered.
C. **R** is reflexive.
D. **R** is an equivalence relation.
E. Only 1 weakly connected component in a directed graph that represents **R**.

注意:背面有試題

7. Suppose the set of "valuable" problems is V, where each problem can be considered by human beings, and there is a bijection between V and **R** (real number). Suppose the set of all A.I. softwares is S, where each software can solve exactly one problem. Each software can be represented as a long binary string (machine code). Which of the following statement are true?
   A. There is a bijection between S and **R**.
   B. A.I. can solve all valuable problems, it just takes a long time to do it.
   C. Without other restrictions, |S| is ∞.
   D. |V|≠|S|, and |V| > |S|.
   E. In terms of solving problems, A.I. may replace human beings someday.

8. The number of ways to buy *n* dollars of tickets is represented by a_n, if only 1-dollar and 2-dollar bills can be used. What of the following can be the recurrence relation for our question? (initial condition: $a_0 = 1; a_1 = 1;$)
   A. $a_n = a_{n-1} + 1$.
   B. $a_n = a_{n-2} + 3$.
   C. $a_n = 2a_{n-2} + 1$.
   D. $a_n = a_{n-1} + a_{n-2}$.
   E. none of the above.

9. About algorithm complexity, which of the following claims are true?
   A. merge sort is $\theta(n \log n)$.
   B. Euclid's algorithm to find gcd(*a*,*b*) is $\theta(\log \max(a, b))$.
   C. bubble sort is $\theta(n \log n)$.
   D. Roy-Warshall Algorithm to compute transitive closure is $\theta(n^2)$ (bit operations).
   E. traveling sales problem is a NP-hard problem.

此題送分

10. Solving the recurrence relation $a_n = a_{n-1} + 20a_{n-2}, a_0 = -1, a_1 = -1, \forall n \geq 2$, using generating function *G(z)*, which following parts are true?
   A. $G(Z) = (1 + z)/(1 - z - 20z^2)$
   B. $G(Z) = (-1 - z)/(1 - z - 20z^2)$
   C. $G(Z) = \left(\frac{-6}{1-5z}\right) + \left(\frac{5}{1+4z}\right)$
   D. $G(Z) = \left(\frac{-2/3}{1-5z}\right) + \left(\frac{-1/3}{1+4z}\right)$
   E. $a_n = \left(\frac{-2}{3} \cdot 5^n\right) + \left(\frac{-1}{3} \cdot (-4)^n\right)$

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線性代數複選題（每題1分）-第十一題到第十五題為複選題（全對才給分）：每一題答對給 1 分、答錯倒扣 1 分，倒扣至線性代數複選題整大題 0 分為止。（5分）

Give the necessary conditions for matrix A can be factorized （複選）

11. $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$.
12. $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$.
13. $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{T}$.
14. $\boldsymbol{A} = \lambda_1 \boldsymbol{u}_1 \boldsymbol{u}_1^T + \lambda_2 \boldsymbol{u}_2 \boldsymbol{u}_2^T + .. + \lambda_n \boldsymbol{u}_n \boldsymbol{u}_n^T$.
15. $\boldsymbol{A} = \boldsymbol{R}^T \boldsymbol{R}$ (Cholesky factorization)

(A) square but non-symmetric, (B) square and symmetric, (C) linearly-independent columns,
(D) linearly-independent eigenvectors, (E) positive definite

第16題到第35題為線性代數單選題：每一題答對給1分、答錯倒扣1分，倒扣至線性代數單選題整大題０分為止。（20分）

If $n \times n$ matrix A is diagonalizable, then

16. 0 is not the eigenvalue of A. (A) True. (B) False.
17. A has n distinct eigenvalues. (A) True. (B) False.
18. A is invertible. (A) True. (B) False.
19. A^T is also diagonalizable. (A) True. (B) False.
20. The diagonalization is unique. (A) True. (B) False.

Suppose $\beta = \{b_1, b_2\}$ is a basis for V and $C = \{c_1, c_2, c_3\}$ is a basis for W. Let $T: V \to W$ be a linear transformation with the standard matrix A for T. If

$$\beta = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}, \quad C = \left\{ \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \right\}, \text{ and } \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Find the transformation matrix M for T relative to β and C.

21. 1, 2, 4 are in the M matrix. (A) True. (B) False.
22. 1, -2, 4 are in the M matrix. (A) True. (B) False.
23. 1, -2, -4 are in the M matrix. (A) True. (B) False.

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24. 7, 14, 28 are in the M matrix. (A) True. (B) False.
25. 7, -14, 28 are in the M matrix. (A) True. (B) False.

If $\boldsymbol{A}$ is a $m \times n$ matrix; $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is an inconsistent system and has a least-square solution $\hat{\boldsymbol{x}}$.

26. $\boldsymbol{b}$ is exactly not in Col $\boldsymbol{A}$. (A) True. (B) False.
27. $\hat{\boldsymbol{x}}$ is not a unique solution. (A) True. (B) False.
28. If $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is not invertible, then the system has no solution. (A) True. (B) False.
29. $\boldsymbol{A}\hat{\boldsymbol{x}}$ is not always in Col $\boldsymbol{A}$. (A) True. (B) False.
30. $\boldsymbol{b} - \boldsymbol{A}\hat{\boldsymbol{x}}$ is always in Nul $\boldsymbol{A}^{\mathrm{T}}$. (A) True. (B) False.

If $\boldsymbol{C}$ is a $n \times n$ covariance matrix, then

31. $\boldsymbol{C}$ always have n orthogonal eigenvectors. (A) True. (B) False.
32. $\boldsymbol{C}$ always have n distinct real eigenvalues. (A) True. (B) False.
33. $\boldsymbol{C}$ has positive and negative eigenvalues but no zero eigenvalues. (A) True. (B) False.
34. $\boldsymbol{C}$ can always be decomposed into $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}$, where $\boldsymbol{A}$ is a matrix. (A) True. (B) False.
35. $\boldsymbol{C}$ is exactly positive definite. (A) True. (B) False.

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線性代數多選題，每題 5 分，每選項單獨計分，答對每一選項的分數為 1 分，答錯每一選項倒扣 1 分，倒扣至線性代數多選題整大題 0 分為止。(25分)

36. Which of the following statements regarding the cost of methods for solving an $n \times n$ linear system $\boldsymbol{Ax} = \boldsymbol{b}$ are true?
  A. The cost of computing the LU factorization is generally proportional to n^2.
  B. The cost of backward substitution on a dense upper triangular matrix is generally proportional to n^2.
  C. If a matrix $\boldsymbol{A}$ has no more than 3 non-zero entries per row, the cost of each iteration of the Jacobi method is proportional to n.
  D. If a matrix $\boldsymbol{A}$ has no more than 3 non-zero entries per row, the cost of each iteration of the Jacobi method is proportional to n^3.
  E. None.

37. Which of the following methods can be used for solving the system $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A}$ is a symmetric, diagonally dominant, square $n \times n$ matrix?
  A. LU factorization with full pivoting.
  B. System of normal equations.
  C. Gauss-Seidel method.
  D. Jacobi method.
  E. None.

38. Which set of vectors is a basis for $\mathbf{R}^3$
  A. {[1,2,3], [-1, 0, 1], [4, 9, 7]}
  B. {[1, 4, 7], [2, 5, 8], [3, 6, 9]}
  C. {[1, 0, -1], [2, 1, -1], [-2, 1, 4]}
  D. {[4, 0, 0], [0, 0, 5], [0, 3, 0]}
  E. All

39. Which vectors form the basis of null(**A**)?

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$$

  A. [-3, 2, 1, -1, 1]
  B. [-1, -2, 1, 0, 0]
  C. [1, -3, 0, -4, 1]
  D. [2, -1, 0, 1, -1]
  E. None

40. A python function Fun is designed to perform a certain matrix operation:

```
def FunSub(m,i,j):
    return [row[:j] + row[j+1:] for row in (m[:i]+m[i+1:])]

def Fun(m):
    if len(m) == 2:
        return m[0][0]*m[1][1]-m[0][1]*m[1][0]

    x = 0
    for c in range(len(m)):
        x += ((-1)**c)*m[0][c]*Fun(FunSub(m,0,c))
    return x
```

What inputs make Fun output 0?

A. $\begin{bmatrix} 7 & 9 & 9 \\ 3 & 2 & 2 \\ 5 & 3 & 3 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 3 & 6 & 7 \\ 8 & 14 & 2 & 1 \\ 3 & 6 & 9 & 12 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 3 & 6 & 7 \\ 8 & 14 & 2 & 1 \\ 3 & 6 & 9 & 18 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 2 & 9 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

E. All