- 本測驗試題為多選題（答案可能有一個或多個），請選出所有正確或最適當的答案，並請用2B鉛筆作答於答案卡。
- 共二十題，每題五分。每題ABCDE每一選項單獨計分；每一選項的個別分數為一分，答錯倒扣一分，倒扣至本份試題0分為止。

Notation: In the following questions, $\mathbb{R}$ is the usual set of all real numbers. We will use underlined letters such as $\underline{a} \in \mathbb{R}^n$ to denote a real, column vector $\underline{a}$ of length n and similarly will use boldface letters such as $\mathbf{A} \in \mathbb{R}^{m\times n}$ to denote a real matrix $\mathbf{A}$ of size $(m \times n)$. $\underline{0}$ is the all-zero column vector of proper length. $\mathbf{A}^\top$ means the transpose of matrix $\mathbf{A}$. rank($\mathbf{A}$) denotes the rank of matrix $\mathbf{A}$. $\mathbf{I}_n$ is the $(n \times n)$ identity matrix. $\|\underline{a}\|$ means the Frobenius norm of vector $\underline{a}$. det($\mathbf{A}$) and tr($\mathbf{A}$) are respectively the determinant and trace of square matrix $\mathbf{A}$. row($\mathbf{A}$) and col($\mathbf{A}$) are the row and column spaces of $\mathbf{A}$ over $\mathbb{R}$, respectively. Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$; then by $\mathcal{W}^\perp$ we mean the orthogonal complement of $\mathcal{W}$ in the Euclidean inner product space $\mathbb{R}^n$. By span$\{\underline{w}_1, \ldots, \underline{w}_k\}$ we mean the vector space generated by vectors $\underline{w}_1, \ldots, \underline{w}_k$ over $\mathbb{R}$. $\mathscr{L} : f(t) \mapsto F(s)$ and $\mathscr{L}^{-1} : F(s) \mapsto f(t)$ denote the unilateral Laplace and inverse Laplace transforms for $t \geq 0$, respectively.

1、 Let

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

be the reduced row echelon form of a matrix $\mathbf{A} = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 & \underline{a}_4 & \underline{a}_5 \end{bmatrix}$. Which of the following statements is/are true?

(A) $\underline{a}_2$ is the 4×1 zero vector.

(B) The three column vectors $\underline{a}_1$, $\underline{a}_4$ and $\underline{a}_5$, are linearly independent.

(C) The column space of $\mathbf{R}$ is the same as the column space of $\mathbf{A}$.

(D) Let $\mathbf{B} = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \end{bmatrix}$ be a submatrix of $\mathbf{A}$. The reduced row echelon form of $\mathbf{B}$ is not always a submatrix of $\mathbf{R}$.

(E) None of the above is true.

2、 Continue from Question 一. Which of the following statements is/are true?

(A) The nonzero rows of $\mathbf{R}$ form a basis for $\mathrm{col}(\mathbf{A}^\top)$.

(B) Let $\mathbf{C}$ be the reduced row echelon form of $\mathbf{R}^\top$. Then $\mathbf{C}^\top\mathbf{C}$ is the identity matrix.

(C) Let $\mathbf{E}$ be a 3×5 matrix and

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

be the reduced row echelon of $\mathbf{E}$. We can obtain the intersection of the right null space of $\mathbf{A}$ and the right null space of $\mathbf{E}$ from $\mathbf{R}$ and $\mathbf{D}$ without knowing $\mathbf{A}$ and $\mathbf{E}$.

(D) Let $\mathbf{Q}$ be a 5×5 matrix and $\mathrm{rank}(\mathbf{Q}) = 3$. Then $\mathrm{rank}(\mathbf{AQ}) = 3$.

(E) None of the above is true.

3、 Let $\mathcal{U}$ and $\mathcal{V}$ be subspaces of $\mathbb{R}^n$ and T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Which of the following sets is/are subspace(s) of $\mathbb{R}^n$?

(A) $\{\underline{a} + \underline{b} \in \mathbb{R}^n : \underline{a} \in \mathcal{U} \text{ and } \underline{b} \in \mathcal{V}\}$.

(B) $\{\underline{a} \in \mathbb{R}^n : \underline{a} \in \mathcal{U} \text{ or } \underline{a} \in \mathcal{V}\}$.

(C) $\{\underline{a} \in \mathbb{R}^n : \underline{a} \in \mathcal{U} \text{ and } \underline{a} \in \mathcal{V}\}$.

(D) $\{\underline{a} \in \mathbb{R}^n : T(\underline{a}) = \underline{0}\}$.

(E) None of the above is true.

4、Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be $n \times n$ matrices for some positive integer n. Which of the following statements is/are true?

(A) $\det(\mathbf{A}) = 0$ implies $\det(\mathbf{R}) = 0$, where $\mathbf{R}$ is the reduced row echelon form of $\mathbf{A}$.

(B) If $\mathbf{AB} = \mathbf{CA} = \mathbf{I}_n$, then $\mathbf{B} = \mathbf{C}$.

(C) If $\det(\mathbf{AB}) \neq 0$, then $\mathbf{A}$ is invertible.

(D) $\text{rank}(\mathbf{AB})=\text{rank}(\mathbf{BA})$.

(E) None of the above is true.

5、Let $\mathcal{V}$ be the vector space of all $\mathbf{A} \in \mathbb{R}^{m\times p}$, with the operations of matrix addition and multiplication of a matrix by a real scalar. Let T be a linear transformation from $\mathcal{V}$ to $\mathcal{V}$, and $\{\mathbf{B}_1, \mathbf{B}_2, \cdots, \mathbf{B}_n\}$ be a basis for $\mathcal{V}$. Suppose U is a linear transformation from $\mathcal{V}$ to $\mathbb{R}^n$ given by $U(c_1\mathbf{B}_1 + c_2\mathbf{B}_2 + \cdots + c_n\mathbf{B}_n) = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}^\top$. Which of the following statements is/are true?

(A) Let $\{\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_k\}$ be linearly independent over $\mathbb{R}$. The set $\{U(\mathbf{A}_1), U(\mathbf{A}_2), \cdots, U(\mathbf{A}_k)\}$ can be linearly dependent over $\mathbb{R}$.

(B) Suppose the dimension of the range space of T is k over $\mathbb{R}$. Then $\{T(\mathbf{B}_1), T(\mathbf{B}_2), \cdots, T(\mathbf{B}_k)\}$ are linearly independent over $\mathbb{R}$.

(C) $n = m$.

(D) Let $\underline{c} = U(\mathbf{B}_1)$ and $\underline{a} = U(T(\mathbf{B}_1))$, then $\underline{a}^\top \underline{a} = \underline{c}^\top \underline{c}$.

(E) None of the above is true.

6、 For the matrix

$$\mathbf{A} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$$

which of the following statements is/are true?

(A) $\text{rank}(\mathbf{A}) = 3$.

(B) The sum of eigenvalues of $\mathbf{A}$ is 6.

(C) $\mathbf{A}$ is similar to $\mathbf{B} = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

(D) The system of linear equations $\mathbf{A}\underline{x} = [-1\ 1\ 1]^\top$ has a solution.

(E) None of the above is true.

7、 For a non-zero matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, define

$$\sigma = \max_{\|\underline{x}\|=1} \|\mathbf{A}\underline{x}\| \tag{1}$$

the maximum Frobenius norm $\|\mathbf{A}\underline{x}\|$ over all unit-norm vectors $\underline{x}$. Which of the following statements is/are true?

(A) σ^2 is the maximal eigenvalue of $\mathbf{A}\mathbf{A}^\top$.

(B) If $\underline{x}_0$ is an optimal solution to equation (1), i.e., $\|\mathbf{A}\underline{x}_0\| = \sigma$, then there exists a vector $\underline{x}_1$ such that $\mathbf{A}\underline{x}_1 = \underline{0}$ and $\underline{x}_1^\top \underline{x}_0 \neq 0$.

(C) Assume $m = n$ and $\mathbf{A}$ is nonsingular. Then $\frac{1}{\sigma} = \max_{\|\underline{x}\|=1} \|\mathbf{A}^{-1}\underline{x}\|$.

(D) Assume $m = n$. Then $\mathbf{I}_n + \mathbf{A}$ is singular only if $\sigma \geq 1$.

(E) None of the above is true.

注意:背面有試題

8、Let $\mathcal{V}$ together with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ be an inner product space over $\mathbb{R}$. For a collection of N linearly independent vectors $\{\underline{w}_1,\ldots,\underline{w}_N\}$ in $\mathcal{V}$, let $\mathbf{M}=[m_{i,j}]\in\mathbb{R}^{N\times N}$ be the corresponding Gram matrix, i.e., the (i,j)-th entry of $\mathbf{M}$ is given by $m_{i,j}=\langle\underline{w}_i,\underline{w}_j\rangle_{\mathcal{V}}$. Define the following set of real-valued functions $f:\mathcal{V}\to\mathbb{R}$

$$\mathcal{F}:=\left\{f(\underline{x})=\sum_{n=1}^{N}\alpha_n\langle\underline{w}_n,\underline{x}\rangle_{\mathcal{V}}:\alpha_1,\ldots,\alpha_N\in\mathbb{R}\right\}$$

which is a vector space over $\mathbb{R}$ when combined with standard addition and scalar multiplication of real-valued functions. For any $f(\underline{x})=\sum_{n=1}^{N}\alpha_n\langle\underline{w}_n,\underline{x}\rangle_{\mathcal{V}}$ and $g(\underline{y})=\sum_{m=1}^{N}\beta_m\langle\underline{w}_m,\underline{y}\rangle_{\mathcal{V}}$ in the vector space $\mathcal{F}$, define

$$\langle f,g\rangle_{\mathcal{F}}=\sum_{n=1}^{N}\sum_{m=1}^{N}\alpha_n\beta_m\langle\underline{w}_n,\underline{w}_m\rangle_{\mathcal{V}}$$

Which of the following statements is/are true?

(A) $\mathbf{M}$ is positive definite.

(B) If $m_{1,1}=2$, $m_{2,2}=1$, and $m_{1,2}=-\frac{1}{2}$, then $\|\underline{w}_1-\underline{w}_2\|_{\mathcal{V}}=4$, where $\|\cdot\|_{\mathcal{V}}$ is the vector norm induced by the inner product $\langle\cdot,\cdot\rangle_{\mathcal{V}}$.

(C) $\langle f,g\rangle_{\mathcal{F}}$ is an inner product for elements $f,g\in\mathcal{F}$.

(D) Given $g(\underline{y})=\langle\underline{w}_1,\underline{y}\rangle_{\mathcal{V}}$, we have $\langle f,g\rangle_{\mathcal{F}}=f(\underline{w}_1)$ for all $f\in\mathcal{F}$.

(E) None of the above is true.

9、Which of the following statements is/are true?

(A) Let $\underline{x}$ and $\underline{y}$ be two non-zero vectors in $\mathbb{R}^n$. The matrix $\underline{x}\,\underline{y}^\top$ is diagonalizable if $\underline{y}^\top\underline{x} \neq 0$.

(B) If $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, then $\mathbf{A} + \mathbf{A}^{-1} - 2\mathbf{I}_n$ is positive semi-definite.

(C) For $\underline{u}, \underline{v} \in \mathbb{R}^n$ such that $\underline{u}^\top\underline{v} \neq 0$, $\mathbb{R}^n = (\text{span}\{\underline{u}\})^\perp \oplus \text{span}\{\underline{v}\}$, where $\oplus$ is the operation of direct sum of two vector spaces.

(D) For the real matrix $\mathbf{Q} = \mathbf{I}_n - \underline{u}\underline{u}^\top$, where $\underline{u} \in \mathbb{R}^n$, we have $\det(\mathbf{Q}) + \text{rank}(\mathbf{Q}) = n$.

(E) None of the above is true.

10、Let $\mathbf{P} \in \mathbb{R}^{n\times n}$ be the orthogonal projection matrix, with respect to the Euclidean inner product, of vectors in $\mathbb{R}^n$ onto $\text{col}(\mathbf{A})$ for some nonzero matrix $\mathbf{A} \in \mathbb{R}^{n\times k}$. Which of the following statements is/are true?

(A) $\text{tr}(\mathbf{P}^\top\mathbf{P}) \geq \text{rank}(\mathbf{P})$.

(B) $\mathbf{P}$ can have an eigenvalue larger than 2.

(C) $\mathbf{P}\,\underline{x} \neq \underline{x}$ for all $\underline{x} \in \mathbb{R}^n$.

(D) If $k < n$ and $\underline{b} \in \mathbb{R}^n$, the system of linear equations $\mathbf{A}\underline{x} = \underline{b}$ has a unique least squares solution.

(E) None of the above is true.

11、Consider the first order differential equation $xy'(x) = y(x) - \sqrt{x^2 + (y(x))^2}$ with initial condition $y(2) = 0$. Which of the following statements is/are true?

(A) It is a linear differential equation for the dependent variable y.

(B) $y(x)$ is a parabolic function of x.

(C) $y'(x) = -x$.

(D) $y''(x) = -\frac{1}{2}$.

(E) None of the above is true.

12、 For the following homogeneous second order linear differential equation

$$(x^2-1)y''(x)-2xy'(x)+2y(x)=0$$

for $x>1$, given one solution $y_1(x)=x$, the other linearly independent solution $y_2(x)$ can then be derived by setting $y_2(x)=v(x)y_1(x)$. Which of the following statements is/are true?

(A) $(x^3+x)v''(x)+2v'(x)=0$.

(B) $v(x)$ with $v'(x)=\frac{x^2}{x^2+1}$ corresponds to one of the possible solutions.

(C) $v(x)$ with $v'(x)=\frac{x^2-1}{x^2}$ corresponds to one of the possible solutions.

(D) $y(x)=1+x+x^2$ is one of the possible solutions.

(E) None of the above is true.

13、 Continue from Question 十二. Solve the following non-homogeneous second order linear differential equation

$$(x^2-1)y''(x) - 2xy'(x) + 2y(x) = x^2 - 1$$

for $x > 1$ using the variation of parameters, i.e., set the particular solution as

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are homogeneous solutions from Question 十二. Which of the following statements is/are true?

(A) $u_1'(x) = -\frac{x^2+1}{x^2-1}$

(B) $u_1'(x) = -(x^2+1)$

(C) $u_2'(x) = \frac{x}{x^2-1}$

(D) $u_2'(x) = x$

(E) None of the above is true.

14、The following second order system

$$\begin{bmatrix} x''(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} -40 & 8 \\ 12 & -60 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{2}$$

can be transformed into an equivalent first order system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

for some real matrix $\mathbf{A}$ by introducing the four functions $x_1(t) = x(t)$, $x_2(t) = x'(t)$, $x_3(t) = y(t)$ and $x_4(t) = y'(t)$. Which of the following statements is/are true?

(A) $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -40 & 0 & 8 & 0 \\ 0 & 0 & 0 & 1 \\ 12 & 0 & -60 & 0 \end{bmatrix}$.

(B) $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -40 & 8 & 0 & 0 \\ 12 & -60 & 0 & 0 \end{bmatrix}$.

(C) -36 is one of the eigenvalues of $\mathbf{A}$.

(D) 8 is one of the eigenvalues of $\mathbf{A}$.

(E) None of the above is true.

15、Continue from Question 十四. Find the particular solution of the second order system in equation (2) with initial conditions $x(0) = 2$, $x'(0) = 12$, $y(0) = 1$ and $y'(0) = 6$. Which of the following statements is/are true?

(A) $x(t) = 2e^{6t}$.

(B) $x(t) = 2\cos(6t) + 2\sin(6t)$.

(C) $y(t) = e^{6t}$.

(D) $y(t) = \cos(6t) + \sin(6t)$.

(E) None of the above is true.

16、Consider the following second order differential equation

$$3t^2y''(t)+\sin(t)y'(t)-\cos(t)y(t)=0.$$

Let $y_1(t)=t^{r_1}\sum_{n=0}^{\infty}a_nt^n$ and $y_2(t)=t^{r_2}\sum_{n=0}^{\infty}b_nt^n$ be two linearly independent Frobenius series solution for $y(t)$ in the above differential equation when $t>0$. Assuming $r_1\geq r_2$ and $a_0=1$, which of the following statements is/are true regarding the solution $y_1(t)$?

(A) $r_1=1$.

(B) $a_1=\frac{1}{2}$

(C) $a_2=-\frac{1}{60}$

(D) $a_3=\frac{1}{1920}$.

(E) None of the above is true.

17、 Continue from Question 十六. Assuming $b_0 = 2$, which of the following statements is/are true regarding the solution $y_2(t)$?

(A) $r_2 = -\frac{1}{6}$.

(B) $b_1 = 0$

(C) $b_2 = -\frac{5}{18}$

(D) $b_3 = \frac{61}{12960}$.

(E) None of the above is true.

18、Consider the following differential equation for the function $y(t)$ defined for $t \geq 0$

$$y'(t) = \frac{2}{t}\left[y(t) \star (\cosh(t)u(t))\right]$$

where $u(t)$ is the usual unit-step (Heaviside) function and where $\star$ is the usual convolutional operation of two functions. Assume $y(t) = 0$ for $t < 0$. Given the initial condition $y(0) = 1$, which of the following statements is/are true regarding the Laplace transform $Y(s) = \mathscr{L}\{y(t)\}$ of solution $y(t)$ in the above equation?

(A) $Y(s = \frac{1}{2}) = -6$

(B) $Y(s = 1) = 1$

(C) $Y(s = 2) = \frac{3}{2}$

(D) $Y(s = 3) = \frac{4}{3}$

(E) None of the above is true.

19、 Continue from Question 十八. Which of the following statements is/are true regarding the solution $y(t)$?

(A) $y'(1) = 1$

(B) $y'(\pi) = 1$

(C) $y''(1) = -1$

(D) $y''(\pi) = 0$

(E) None of the above is true.

20、 Consider the following partial differential equation for the bivariate function $v(x,t)$

$$\frac{\partial}{\partial t}v(x,t) = \frac{\partial^2}{\partial x^2}v(x,t),\ 0 \le x \le 10,\ t \ge 0$$

subject to conditions

$$\frac{\partial}{\partial x}v(x,t)\Big|_{x=0} = \frac{\partial}{\partial x}v(x,t)\Big|_{x=5} = 0, \quad \text{for all } t \ge 0$$
$$v(x,0) = x^2 - 10x, \quad \text{for all } x \text{ with } 0 < x < 10.$$

Express the solution $v(x,t)$ in the following form

$$v(x,t) = \sum_{n=0}^{\infty} [a_n \cos(p_n \pi x) + b_n \sin(q_n \pi x)] \exp\left(-r_n \pi^2 t\right)$$

for some constants $a_n, b_n, p_n, q_n, r_n \in \mathbb{R}$ satisfying $0 \le r_0 < r_1 < r_2 < \cdots$ and $a_n^2 + b_n^2 > 0$ for $n = 1, 2, \ldots$ Which of the following statements is/are true?

(A) $a_0 = -\frac{50}{3}$

(B) $b_1 = \frac{17}{29}$

(C) $p_3 = \frac{9}{10}$

(D) $r_4 = \frac{32}{50}$

(E) None of the above is true.