國立中央大學111學年度碩士班考試入學試題

所別： 數學系 碩士班 數學組(一般生)　　　　共1頁　第1頁

科目： 高等微積分

第 2 到 6 題為證明題需證明過程，無證明過程者不予計分。

Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{R}$ be the set of real numbers.

1. (*20 points*) True or false. (just write down your answer, do not give any reason)

   (1.1) (*4 points*) Every bounded monotonic sequence in $\mathbb{Q}$ is convergent.

   (1.2) (*4 points*) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences of positive real numbers, then

   $$\limsup_{n\to\infty}(x_n \times y_n) \leq (\limsup_{n\to\infty} x_n)(\limsup_{n\to\infty} y_n).$$

   (1.3) (*4 points*) Suppose f and g are uniformly continuous on $\mathbb{R}$, then fg is also uniformly continuous on $\mathbb{R}$.

   (1.4) (*4 points*) In $\mathbb{R}^2$, the existence of all directional derivatives at a point does not imply differentiability.

   (1.5) (*4 points*) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous on $\mathbb{R}^n$ and let B be a bounded subset in $\mathbb{R}^n$. Then $f(B)$ is bounded.

2. (*10 points*) We say that $A \subseteq \mathbb{R}^n$ is compact if every open cover of A has a finite subcover. Show that the set $\{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : \sqrt{\sum_{i=1}^n x_i^2} < 1\}$ is not compact by the definition.

3. (*15 points*) Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$. Prove that f is Riemann integrable on $[a, b]$.

4. (*15 points*) Let $\{a_n\}$ be bounded sequence in $\mathbb{R}$. Prove that $f(x) = \sum_{n=0}^{\infty} \left(\frac{a_n x^n}{n!}\right)^2$ is continuous on $\mathbb{R}$.

5. (*20 points*) A function is said to be of class C^r if the first r derivative exist and are continuous. A function is said to be of class C^∞ if it is of class C^r for all positive integers r. Give an example of a C^∞ function that is not analytic and explain your answer.

6. (*20 points*) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

   $$f(x,y) = \begin{cases} \frac{x^2 y}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) (*10 points*) Show that f is continuous at $(0,0)$.

   (b) (*10 points*) Investigate the differentiability of f at $(0,0)$.