

所別：光電科學研究所碩士班 不分組 科目：應用數學

- (10%) 1. if a vector  $\vec{F}$  is given by

$$\vec{F} = (x^2 + y^2 + z^2)^n (\hat{i}x + \hat{j}y + \hat{k}z),$$

find

- (a)  $\nabla \cdot \vec{F}$ ,  
(b)  $\nabla \times \vec{F}$ ,  
(c)  $\nabla^2 \vec{F}$ ,  
(d) A scalar potential  $\varphi(x, y, z)$  so that  $\vec{F} = -\nabla\varphi$ .  
(e) For what value of the exponent  $n$  does the scalar potential diverge at both the origin and infinity?

- (10%) 2. (a) Complex numbers,  $a + ib$ , with  $a$  and  $b$  real, may be represented by (or, are isomorphic with)  $2 \times 2$  matrices:

$$a + ib \leftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Show that this matrix representation is valid for (i) addition and (ii) multiplication.

- (b) Find the matrix corresponding to  $(a + ib)^{-1}$ .  
(c) Determine the eigenvalues and eigenvectors of the matrix for  $a + ib$ .

- (10%) 3. Evaluate

- (a)  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$ ,  
(b)  $\lim_{x \rightarrow 0} x^{-n} j_n(x)$ , for  $n = 3$ ,

where  $j_n(x)$  is a spherical Bessel function defined by

$$j_n(x) = (-1)^n x^n \left( \frac{d}{x dx} \right)^n \left( \frac{\sin x}{x} \right)$$

- (10%) 4. (a) Find the Fourier series representation of

$$f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ x, & 0 \leq x < \pi. \end{cases}$$

- (b) From your Fourier expansion show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

- (10%) 5. Find the maximum value of the directional derivative of  $\varphi(x, y, z)$ ,

$$\frac{d\varphi}{ds} = \frac{\partial\varphi}{\partial x} \cos \alpha + \frac{\partial\varphi}{\partial y} \cos \beta + \frac{\partial\varphi}{\partial z} \cos \gamma.$$

subject to the constraint

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

參考用

注意：背面有試題

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6. Find the general solution:

(a)  $(y^3 - y^2 \sin x)y' + y^{-1} \cos x = 0$  (7%)

(b)  $(y^3 - x^2 y)y' - xy^2 = 0$  (7%)

(c)  $\frac{1}{y^2 + 1} y' + \frac{2}{x} \tan^{-1} y = \frac{2}{x}$  (7%)

(d)  $\frac{d^2 x}{dt^2} \exp\left(\frac{dx}{dt}\right) = 1$  (7%)

7. Solve

(a)  $y'' - 6y' + 9y = 0$ ;  $y(-1) = 1$ ,  $y'(-1) = 7$ . (7%)

(b)  $x^2 y'' + 3xy' + 37y = 0$ ;  $y(1) = 1$ ,  $y'(1) = 0$ . (7%)

8. At a certain instant 100gm of a radioactive substance are present. After 4 years, 20 gm remain. How much of the substance remains after 8 years? (8%)

參考用