

科目：高等微積分(1001)

校系所組：中大數學系甲組

清大數學系純粹數學組、應用數學組



1. (12%) If g is continuous and nonnegative on $[0, 1]$ with $g(0) = 0 = g(1)$, show that there exists $a, b \in [0, 1]$ such that $g(a) = g(b)$ and $b - a = 1/2$.
2. (12%) If for each n , $|f_n(x)| \leq M_n$, $x \in E \subset \mathbb{R}$, and if f_n converges uniformly on E to f . Show that there is $M > 0$ such that $|f_n(x)| \leq M$ for all n and $x \in E$.
3. (12%) Let $f(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$, is f differentiable at $(0, 0)$? Give your reason.
4. (12%) Find the extrema of $f(x, y, z) = x^4 + y^4 + z^4$, on the circle defined by

$$x^2 + y^2 + z^2 = 1, \text{ and } x + y + z = 1.$$
5. (15%) Consider the space $C([0, 1]) = \{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$ and for $f, g \in C([0, 1])$, define

$$d(f, g) = \int_0^1 |f(x) - g(x)| dx.$$
 - (a) (5%) Show that d is a metric on $C([0, 1])$.
 - (b) (5%) Is the metric space $(C([0, 1]), d)$ complete? Explain!
 - (c) (5%) Is the set $\{f \in C([0, 1]) : f(x) > 0, \text{ for all } x \in [0, 1]\}$ open in the space $(C([0, 1]), d)$? Explain!
6. (15%) Let

$$F(x) = \int_1^\infty t^{x-1} e^{-t} dt, \quad x \in \mathbb{R}.$$
 - (a) (7%) Show that $F(x)$ exists for $x \in \mathbb{R}$.
 - (b) (8%) Show that F is differentiable for $x \in \mathbb{R}$ and find $F'(x)$.
7. (10%) Is there a continuous function from $[0, 1]$ onto $(0, 1)$? Prove your assertion.
8. (12%) Let $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$ denote the unit half-sphere in \mathbb{R}^3 . Evaluate the surface integral over S :

$$\int_S (x + y + z^2) dS,$$

where dS is the area element on S .