科目：高等微積分　　校系所組：中大數學系甲組　交大應用數學系甲組
清大數學系純粹數學組、應用數學組

共七題，滿分 100 分

1. (10 points) Let $A = [1,2) \cup [3,4]$ be a subset of $\mathbb{R}$. Define a function $f : A \mapsto \mathbb{R}$ by
$$f(x) = \begin{cases} 0 & \text{if } x \in [1,2) \\ 1 & \text{if } x \in [3,4]. \end{cases}$$
Prove or disprove that $f$ is continuous on $A$.

2. (15 points) Let $\{x_n\}$ be a convergent sequence in a metric space and $\lim_{n\to\infty} x_n = x$. Prove or disprove that the set $A = \{x_1, x_2, \cdots\} \cup \{x\}$ is compact.

3. (15 points) Let $f$ and $g$ be two non-constant real-valued functions defined in the same neighborhood $N$ of a point $a \in \mathbb{R}^n$. Suppose that $f$ is differentiable at $a$ and that $g$ is continuous at $a$, but not differentiable at $a$. Is it possible that the function $(f \cdot g) : N \mapsto \mathbb{R}$, defined by $(f \cdot g)(x) = f(x)g(x)$ for $x \in N$, differentiable at $a$? Justify your answer.

4. (15 points) Suppose that $f$ is twice differentiable on an interval $I$ containing $a$ and that $f''$ is continuous on $I$. Compute
$$\lim_{h\to 0} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} = ?$$
Give reasons that support your computation.

參考用

5. (15 points) For $n \in \mathbb{N}$ and $x \in [0,1]$, let
$$f_n(x) = \frac{2nx}{1+n^2x^2}.$$
Find the function $f : [0,1] \mapsto \mathbb{R}$ so that $f_n(x) \to f(x)$ pointwisely on $[0,1]$ as $n \to \infty$. Prove or disprove that the convergence is uniform.

6. (15 points) Show that near $(x_1, x_2, y_1, y_2, y_3) = (0, 1, 3, 2, 7)$ we can solve
$$\begin{cases} 2e^{x_1} + x_2 y_1 - 4y_2 + 3 = 0 \\ x_2 \cos x_1 - 6x_1 + 2y_1 - y_3 = 0 \end{cases}$$
uniquely for $(x_1, x_2)$ as functions of $(y_1, y_2, y_3)$ and find the values $\frac{\partial x_1}{\partial y_1}, \frac{\partial x_1}{\partial y_2}, \frac{\partial x_1}{\partial y_3}$ at the point $(y_1, y_2, y_3) = (3, 2, 7)$.

7. (15 points) Evaluate the integral
$$\int_0^3 \int_0^3 [x+y]\, dx dy,$$
where $[t]$ is the greatest integer $\leq t$.