所別：數學系碩士班 乙組(一般生) 科目：數值分析

Instructions: Do all 5 problems. Show your work.

1. **(Nonlinear equations)**

   (20pts) Consider the following system of nonlinear equations

$$\begin{cases} x+y+z &= 3 \\ x^2+y^2+z^2 &= 5 \\ e^x+xy-yz &= 1 \end{cases} \tag{1}$$

   Give a complete algorithm of Newton's method for the nonlinear systems given by (1), using an initial vector $\mathbf{x}^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)})^T$. The algorithm should include some input and output data, and stopping conditions, etc. Also write explicitly down the Jacobian matrices J.

2. **(Numerical linear algebra)** Consider

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

   and

$$b = (1,2)^T.$$

   (a) (10 pts) Compute the condition number of A, $\kappa(A)$, using the maximum norm.

   (b) (10 pts) For any initial guess $\mathbf{x}^{(0)}$, will the Jacobi method converge to the solution of $Ax = b$? Justify your answer using some theory in iterative methods.

3. **(Numerical Differentiation)**

   (a) (10pts) Derive the second-order central difference approximation for the first derivative, including error term, for $h > 0$:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} - \frac{h^2}{6} f'''(\xi), \tag{2}$$

   where $\xi \in (x_0 - h, x_0 + h)$.

   (b) (10pts) Consider the following table obtained by approximating the derivative of $f(x) = e^x$ at $x_0 = 0$ using the equation (2) with step size h approaching zero.

h	$f'(0)$	h	$f'(0)$
10^0	1.17520119364380	10^{-10}	1.00000008274037
10^{-2}	1.00001666674999	10^{-12}	1.00003338943111
10^{-4}	1.00000000166689	10^{-14}	0.99920072216264
10^{-6}	0.99999999997324	10^{-16}	0.55511151231258
10^{-8}	0.99999999392253	10^{-18}	0.00000000000000

   From the table, we find that the numerical derivative has quadratic convergence from $h = 10^0$ down to 10^{-6} as expected. But the error increases for the cases of $h < 10^{-6}$ and finally it does not converge to 1. Explain why this happened and provide some approach to improve the accuracy of the numerical derivative. Note that Taylor series of e^x around $x_0 = 0$: $1 + x + x^2/2! + x^3/3! + ...$ and $f'(0) = 1$

注意：背面有試題

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4. **(Numerical Integration)**

   (20pts) Construct the following quadrature

$$\int_{-2}^{2} f(x)\,dx = Af(-1) + Bf(0) + Cf(1)$$

   that is **EXACT** for the highest possible degree polynomial. For which degree of polynomial is the formula you derived exact.

5. **(Interpolation)** Consider the function $f(x) = \cos(\frac{\pi}{2}x)$

   (a) (5pts) Form the divided difference table for nodes $x_0 = -2, x_1 = -1, x_2 = 1, x_3 = 2$.

   (b) (5pts) Use the table to construct the interpolating polynomial, $p(x)$, on the nodes.

   (c) (5pts) Write the formula for the error, $E(x) = f(x) - p(x)$, involving derivatives of $f(x)$

   (d) (5pts) Derive an upper bound for the error, $\max_{x\in[-2,2]} |f(x) - p(x)|$. Use the approximation $\pi^2 \approx 10$.