

系所別:

數學系

科目:

線性代數

LINEAR ALGEBRA

1. (a) (10 points) Let  $A$  be a diagonal matrix whose diagonal entries are all different. Show that if  $B$  is a matrix such that  $AB = BA$ , then  $B$  is also diagonal.
- (b) (10 points) Which  $n \times n$  matrices  $B$  have the property that  $AB = BA$  for all  $n \times n$  matrices  $A$ ?
2. (a) (10 points) Factor the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = QR,$$

where  $Q$  is an orthogonal  $4 \times 3$  matrix over  $\mathbb{R}$  (i.e.  $Q^T Q = I$ ) and  $R$  is an upper triangular matrix.

- (b) (5 points) Find the projection of  $b = [1, 0, 0, 1]^T$  on the column space of  $A$ .
- (c) (5 points) Find a solution to the problem of minimizing  $\|Ax - b\|$  for all  $x \in \mathbb{R}^3$ .

$$3. \text{ Let } A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

- (a) (10 points) Find a diagonal matrix  $D$  and an orthogonal matrix  $S$  such that  $A = SDS^{-1}$ .
- (b) (10 points) Compute  $A^{20}$ .

4. Suppose that  $S, T$  are subspaces of a finite-dimensional vector space  $V$ .

- (a) (10 points) Show that  $\dim S + \dim T = \dim(S \cap T) + \dim(S + T)$ .
- (b) (10 points) Let  $P_3(\mathbb{R})$  be the space of all polynomials over  $\mathbb{R}$  of degree less than or equal to 3. Let  $S$  be the subspace of  $P_3(\mathbb{R})$  spanned by  $1 - x + x^2, x - x^2 + x^3$  and let  $T$  be the subspace spanned by  $1 + x, x + x^2, x^2 + x^3$ . Find bases for  $S + T$  and  $S \cap T$ .

$$5. (a) (10 points) \text{ Find the Jordan canonical form of } A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

- (b) (10 points) How many possible Jordan forms are there for a  $7 \times 7$  complex matrix with characteristic polynomial  $(x + 1)(x - 2)^4(x + 3)^2$ ?

參考  
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