系所別: 數學系 科目: 抽象代數

以下各題，只給答案，沒有說明，不給分

In the following, the symbols $\mathbb{Q}$ and $\mathbb{C}$ denote the fields of rational numbers and complex numbers as usual.

1. Determine whether or not the following statements is correct. Explain your answers.
   (a) (4 分) For any given positive integer n there exists a finite group of order n.
   (b) (6 分) For any given positive integer n there are only finitely many *non-isomorphic* groups of order n.

2. Let G be a *non-Abelian* (non-commutative) group and let A be a cyclic group. Assume that G has an action $*$ on A which satisfies (1) $(\sigma\tau) * a = \sigma * (\tau * a)$ for all $\sigma, \tau \in G$ and all $a \in A$; and (2) $\sigma * (ab) = (\sigma * a)(\sigma * b)$ for $\sigma \in G$ and $a, b \in A$.
   (a) (8 分) Show that the action $*$ induces a group homomorphism from G to $\mathrm{Aut}(A)$ where $\mathrm{Aut}(A)$ is the automorphism group of A.
   (b) (4 分) Show that there exist a non-trivial normal subgroup H of G of finite index (that is, $\{e\} \neq H \triangleleft G$ and $[G : H]$ is finite) so that $\sigma * a = a$ for all $\sigma \in H$ and all $a \in A$.
   (c) (4 分) Suppose that A is an infinite cyclic group. Show that the index $[G : H]$ of H in G is either one or two.

參考用

3. Let p be a prime number and let S_p be the symmetric group on p symbols.
   (a) (10 分) Determine the number of p-Sylow subgroups of S_p (*Hint*: first show that S_p has $(p-1)!$ elements of order p).
   (b) (8 分) What are the numbers of p-Sylow subgroups of S_{p+i} for any i such that $1 \leq i \leq p-1$? You need to explain your answer.

4. Let R be a finite ring.
   (a) (7 分) Can R be an integral domain if R has order $|R| = 36$? Why?
   (b) (7 分) What should be a necessary condition for the order of R so that R can be an integral domain? Explain your answer.

5. Let F be a field. Let $f_1(x), f_2(x), \ldots, f_n(x) \in F[x]$ be polynomials which are not all zero. The G.C.D. of $f_1(x), f_2(x), \ldots, f_n(x)$ is defined to be the monic polynomial of

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maximal degree among common divisors of $f_1(x), f_2(x), \ldots, f_n(x)$. Denote the G.C.D. by $\gcd(f_1(x), f_2(x), \ldots, f_n(x))$. Prove

$$\gcd(f_1(x), f_2(x), \ldots, f_n(x)) = \sum_{i=1}^{n} a_i(x) f_i(x) \text{ for some } a_i(x) \in F[x],\ i = 1, \ldots, n$$

by completing the following steps.

(a) (12 %) Show that $F[x]$ is a principal ideal domain.

(b) (8 %) Let $\mathcal{A}$ be the ideal generated by $f_1(x), f_2(x), \ldots, f_n(x)$. Show that $\mathcal{A}$ is also generated by $\gcd(f_1(x), f_2(x), \ldots, f_n(x))$ and conclude that

$$\gcd(f_1(x), f_2(x), \ldots, f_n(x)) = \sum_{i=1}^{n} a_i(x) f_i(x).$$

6. (10 %) Let a, b be relatively prime non-zero integers such that at least one of a, b is not ± 1. Assume that both a and b are square free integers. Is it true that the polynomial $ax^m - b$ is irreducible in $\mathbb{Q}[x]$ for every positive integer m? Explain your answer.
Note. An integer n is called square free if it has that property that for prime p with $p \mid n$ then $p^2 \nmid n$.

7. (12 %) Let $P(x)$ be an irreducible polynomial in $\mathbb{Q}[x]$. Let $r \in \mathbb{C}$ be a root of $P(x)$. Given

$$\alpha = \frac{a_n r^n + \cdots + a_1 r + a_0}{b_m r^m + \cdots + b_1 r + b_0} \text{ where } a_i, b_j \in \mathbb{Q} \text{ and } b_m r^m + \cdots + b_1 r + b_0 \neq 0.$$

Prove or disprove that there exists a polynomial $f_\alpha(x) \in \mathbb{Q}[x]$ such that $\alpha = f_\alpha(r)$.