1. (25%)
Solve the following ordinary differential equations.

(a) $y'' - 4y' + 13y = \delta(t-1)$, $y(0) = 0$, $y'(0) = 3$,
$\delta(t)$ is the unit impulse function. (6%)

(b) $x^2 y'' + 3xy' - 15y = x^2 e^x$ (6%)

(c) $x^2 y'' + xy' + x^2 y - 16y = 0$ (4%)

(d) $dx/dt + 4x + dy/dt = 1$ (9%)
$dx/dt - 2x + y = t^2$, $x(0) = 2$ and $y(0) = -1$

2. (10%)
Assume that a function $f(x)$, defined on the interval $(0,1)$, can be represented by a series of Bessel functions, i.e., assume $f(x) = c_0 J_k(r_0 x) + c_1 J_k(r_1 x) + c_2 J_k(r_2 x) + \ldots\ldots$, where $r_0, r_1, r_2, \ldots$, are the distinct, positive zeros of $J_k(x)$ and k is a fixed real number. How to determine the coefficients of $c_0, c_1, c_2, \ldots$.

3. (10%)
If $A = (3x^2 + 6y)\,i - 14yz\,j + 20xz^2\,k$, evaluate $\int A \cdot dr$ from (0,0,0) to (1,1,1) along the following path C:
$x = t$, $y = t^2$, $z = t^3$

4. (15%)
Consider the matrix

$$A = \begin{vmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$$

(a) Compute the eigenvalues and eigenvectors.
(b) Compute A^4 using the Hamilton-Cayley theorem.

5. (20%)
The cooling of a semi-infinite slab can be described by

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial z^2}\text{, @ } t=0\text{, for all z: } \theta = 1\text{, @ } z=0\text{, for all } t>0\text{: } \frac{\partial \theta}{\partial z} = \frac{h}{k}\theta,$$

where θ is a dimensionless temperature, α the thermal diffusivity, h heat transfer coefficient, and k thermal conductivity. Please solve this equation (Hint: perform a transformation first to obtain a homogeneous boundary condition).

6. (20%)
Solve the PDE problem
$u_t = \alpha^2 u_{xx} - \beta u \quad 0 < x < 1,\ 0 < t < \infty$
where α and β are constants
with boundary conditions
$u(0,t) = 0$
$u(1,t) = 0 \qquad 0 < t < \infty$
and initial condition
$u(x, 0) = \phi(x) \quad 0 \le x \le 1$