所別：通訊工程學系碩士班 乙組 科目：工程數學

1. (10 %) Consider the system of linear equations $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & k+1 & 6 \\ -1 & 3 & k-2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 8 \\ -1 \end{bmatrix}$$

Determine the values of k such that:

(a) The system has infinitely many solutions.

(b) The system has a unique solution.

(c) The system has no solution.

2. (10 %) Let T : $\mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the linear transformation given by

$$\mathrm{T}(\mathbf{A}) = \mathbf{A}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{A}$$

Find a basis for the kernel (nullspace) and a basis for the image of T.

3. (10 %) Let **A** be the matrix $\mathbf{A} = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}$.

   (a) Find an invertible matrix **P** and a diagonal matrix **D** such that $\mathbf{A} = \mathbf{PDP}^{-1}$.

   (b) Compute $\mathbf{A}^{100}$.

   (c) Find a square matrix **B** such that $\mathbf{B}^5 = \mathbf{A}$.

4. Give the correct choice of the following statements.

(a) (5 %) Let **V** and **W** be subspace of $\mathbb{R}^5$ such that dim(**V**) = 4 and dim(**W**) = 2. Then the possible dimension for $\mathbf{V} \cap \mathbf{W}$ is

   (i) 0,1 (ii) 1,2 (iii) 2,3 (iv) none of above

(b) (5 %) If $\{u_1, \cdots, u_m\} \subset \mathbb{R}^4$ is linear independent and $\{v_1, \cdots, v_l\}$ spans $\mathbb{R}^4$, then

   (i) $l, m \ge 4$ (ii) $l, m \le 4$ (iii) $l \ge 4, m \le 4$ (iv) $l \le 4, m \ge 4$ (v) none of the above

(c) (5 %) If 3 is an eigenvalue of **A**, then $\mathbf{A}^2 - 4\mathbf{A} + 3\mathbf{I}$ is invertible (i) True (ii) False (答錯倒扣 5%)

(d) (5 %) The matrix $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are similar. (i) True (ii) False (答錯倒扣 5%)

5. In Fig 1, the random variable X has two possible values +1 and −1, and X is added by a uniform random variable N over (-2,2) to form a new variable. The value of the variable Y is decided by passing the new variable through a sign-function: {Y|Y=+1 if X+N≥0 and Y=−1 if X+N<0}. Assume that the occurrence probabilities of X=+1 and X=−1 are 0.6 and 0.4, respectively.

參考用

注意：背面有試題

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(a) (6%) What is the probability of X=−1 when Y=+1?

(b) (6%) To reduce the probability in (1), we enlarge the amplitude of X more than unity. What is the minimal amplitude of X such that the probability is below 0.1?

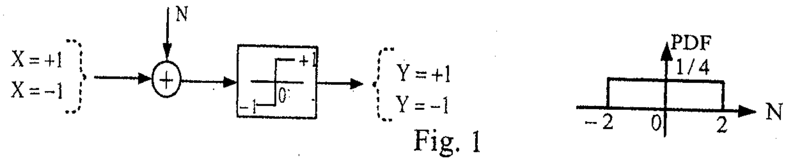

Fig. 1

6. In Fig. 2, X and Y are two i.i.d. Gaussian random variables, $N(0,\sigma^2)$. V and W are obtained by a linear combination of X and Y. Z is obtained by passing V and W through square-law devices, adder, and a square-root device.
   (a) (7%) Prove that V and W are statistically independent.
   (b) (7%) Find the PDF of Z.

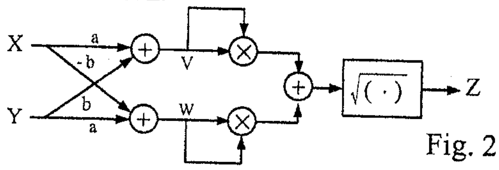

Fig. 2

7. A random variable X is defined by

$$f_X(x) = 4e^{-2|x|}.$$

The random variable Y is related to X by Y=2X−3.
   (a) (6%) Use the characteristic function to determine E[X] and E[X^2].
   (b) (6%) Find E[Y], E[Y^2], and σ_Y^2.

8. Given a random variable X with the PDF

$$f_X(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2} u(x),$$

where u(x) is the unit step function, {u(x)|u(x)=1 if $x \geq 0$ and u(x)=0 if x<0}, and the random variable $Y = e^{-X^2}$. We use $\hat{Y} = aX + b$ to approach Y, where a and b are two constants. Define the mean squared error (MSE) as

$$e_{MSE} = E[(Y-\hat{Y})^2] = E[(Y-aX-b)^2].$$

Then we can treat e_{MSE} as a quadratic function of a and b.
   (1) (4%) By taking the derivative of e_{MSE} with respect to a and b, find the necessary condition to minimize e_{MSE}.
   (2) (8%) Find the values of a and b for the best estimate $\hat{Y} = aX + b$. [Note: You can use the numerical approximation $\sqrt{2\pi} \approx 2.5$.]