國立中央大學 109 學年度碩士班考試入學試題

所別：　數學系 碩士班　數學組(一般生)
　　　　數學系 碩士班　應用數學組(一般生)
　　　　數學系 碩士班　應用數學組(在職生)

科目：　線性代數

本科考試禁用計算器　　　　　　　　*請在答案卷(卡)內作答

Instructions: Do all problems. Show your work.

1. Let $A = \begin{pmatrix} -3 & -1 & 1 \\ -1 & -3 & 1 \\ -2 & -2 & 0 \end{pmatrix}$.

   (a) Find the characteristic polynomial of A. (5 pts)

   (b) Explain why A is diagonalizable or not diagonalizable. (5 pts)

   (c) Find a Jordan canonical form J of A. (5 pts)

   (d) Find a matrix Q such that $Q^{-1}AQ = J$. (10 pts)

2. Consider the line L spanned by the vector $w = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^4$. Let $\mathrm{Proj}_L : \mathbb{R}^4 \to \mathbb{R}^4$ be the function that sends a vector v to its orthogonal projection onto L.

   (a) By definition, $\mathrm{Proj}_L(v) = kw$ for some scalar k, express this scalar in terms of v and w. (5 pts)

   (b) Using your answer from (a) to prove that Proj_L is a linear transformation. (5 pts)

   (c) Write down the matrix of Proj_L. (5 pts)

3. Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation given by

$$Tv = \begin{pmatrix} 1 & 2 & 3 & 0 & 1 \\ 1 & 3 & 1 & 1 & 2 \\ 2 & 6 & 2 & 2 & 4 \\ 3 & 6 & 6 & 0 & 3 \end{pmatrix} v, \quad \text{where } v \in \mathbb{R}^5.$$

   Find a basis for the null space of T and the dimension of the range. (10 pts)

注意:背面有試題

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4. (a) Prove that if $n \times n$ matrices A and B are invertible, then the product AB is also an invertible matrix. (5 pts)

   (b) Prove that the determinant of an $n \times n$ skew-symmetric matrix is zero if n is odd. (5 pts)

5. Apply the Gram-Schmidt orthogonalization process to

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

and write the result in the form $A = QR$, where $A = [v_1, v_2, v_3]$, Q is an orthogonal matrix and R is an upper triangular matrix. (10 pts)

6. Let U and V are finite dimensional vector spaces over a scalar field $\mathbb{F}$. Consider a linear transformation $T : U \to V$. Prove that if $\dim U > \dim V$, then T cannot be injective. (10 pts)

7. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with n distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and corresponding unit eigenvectors $u_1, u_2, \cdots, u_n$. Denote $P_i = u_i u_i^T$ for $1 \leq i \leq n$. Show that

   (a) u_i is orthogonal to u_j for $i \neq j$. (5 pts)

   (b) $\sum_{i=1}^{n} P_i = I$. (5 pts)

8. Let $n > 1$ be a positive integer. Let $V = M_{n\times n}(\mathbb{C})$ be the vector space over $\mathbb{C}$ consists of all complex $n \times n$ matrices. Prove that, for any $A \in V$, the set

$$S_A = \left\{ I = A^0, A, A^2, \cdots, A^{n^2-1} \right\}$$

cannot be a basis of the vector space V. (10 pts)

參考用

注意：背面有試題