國立中央大學110學年度碩士班考試入學試題

所別： 數學系 碩士班 數學組(一般生)　　　　共1頁　第1頁

科目： 高等微積分

本科考試禁用計算器　　　　＊請在答案卷(卡)內作答

第(2)至(6)題為證明題需證明過程，無證明過程者不予計分

Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{R}$ be the set of real numbers.

(1) (*20 points*) True or false? (just write down your answer, do not give any reason)

(1.1) (*2 points*) Every Cauchy sequence in $\mathbb{Q}$ is convergent.

(1.2) (*2 points*) If $\{x_n\}$ and $\{y_n\}$ are bounded sequences in $\mathbb{R}$, then

$$\limsup_{n\to\infty}(x_n \times y_n) \leq (\limsup_{n\to\infty} x_n)(\limsup_{n\to\infty} y_n).$$

(1.3) (*2 points*) Every compact set in the metric space is closed and bounded set.

(1.4) (*2 points*) If A is connected in $\mathbb{R}^n$, then $\mathbb{R}^n \backslash A$ is also connected.

(1.5) (*2 points*) The function $f(x) = x$ is continuous on $[0,1] \cup \{2\}$.

(1.6) (*2 points*) Let f be a continuous real function on $\mathbb{R}$ and let $Z(f)$ be the set of all $p \in \mathbb{R}$ at which $f(p) = 0$. Then $Z(f)$ is closed.

(1.7) (*2 points*) The function $f(x) = \frac{1}{x}$ is uniformly continuous on $(0,1)$.

(1.8) (*2 points*) Let f be a real function on $[0,2]$, then f is Riemann integrable on $[0,2]$ if and only if $|f|$ is Riemann integrable on $[0,2]$.

(1.9) (*2 points*) Every differentiable function on $[0,2]$ is Riemann integrable.

(1.10) (*2 points*) Let $\{P_n\}_{n\in\mathbb{N}}$ a uniformly convergent sequence of polynomials on $[0,2]$ and $f = \lim_{n\to\infty} P_n$. Then f is differentiable.

(2) (*10 points*) Let $B = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < 1\}$. Show that B is open set.

(3) (*15 points*) Let (M,d) be metric space and $\{x_n\}_{n=1}^{\infty} \subseteq M$ converge to $x \in M$. Show that the set $\{x_1, x_2, x_3, \cdots\} \cup \{x\}$ is compact set.

(4) (*20 points*) Every rational x can be written in the form $x = \frac{m}{n}$, where $n > 0$, and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{R}\backslash\mathbb{Q} \\ \frac{1}{n} & \text{if} \quad x = \frac{m}{n} \end{cases}.$$

Prove that f is continuous at every irrational point, and that f has a discontinuity at every rational point.

(5) (*15 points*) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous on $\mathbb{R}^n$ and let B be a bounded subset in $\mathbb{R}^n$. Prove or disprove (if you think the following statement is false, give a counter-example and prove that your example works) that $f(B)$ is bounded.

(6) (*20 points*) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) (*10 points*) Show that f is continuous at $(0,0)$.

(b) (*10 points*) Investigate the differentiability of f at $(0,0)$.