Problem 1. **(10 %)** Given a matrix

$$\bar{\bar{\mathbf{A}}}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 3 & 0 & a \end{bmatrix}, \tag{1}$$

where a is a real number. Find the *rank* of $\bar{\bar{\mathbf{A}}}_1$ and give the corresponding range of a.

Problem 2. **(15 %)** For the matrix

$$\bar{\bar{\mathbf{M}}}_1 = \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}, \tag{2}$$

(i) (10 %) Find the *Eigenvalues* and *Eigenvectors* for $\bar{\bar{\mathbf{M}}}_1$.

(ii) (5 %) Show that there exists a matrix $\bar{\bar{\mathbf{J}}}$, which is similar to the matrix $\bar{\bar{\mathbf{M}}}_1$ in the **Jordon form**, i.e.,

$$\bar{\bar{\mathbf{J}}} = \bar{\bar{\mathbf{Q}}}^{-1}\,\bar{\bar{\mathbf{M}}}_1\,\bar{\bar{\mathbf{Q}}}. \tag{3}$$

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Problem 3. (25 %) Let $\mathbb{R}^{m\times n}$ and $\mathbb{R}^n$ denote the set of real $m \times n$ matrices and the set of real $n \times 1$ column vectors, respectively, and let $\mathbb{S}^n_+$ denote the set of real $n \times n$ symmetric positive semidefinite (PSD) matrices. This problem includes two parts as follows:

(a) (15 %) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ with r singular values $\sigma_1, \ldots, \sigma_r$, $\mathbf{U}_r = [\mathbf{u}_1, \ldots, \mathbf{u}_r]$ consisting of the associated r left singular vectors and $\mathbf{V}_r = [\mathbf{v}_1, \ldots, \mathbf{v}_r]$ consisting of the associated r right singular vectors.

(i) Find the range space (i.e., column space) of $\mathbf{A}$, denoted as $\mathcal{R}(\mathbf{A})$, and the row space of $\mathbf{A}$ (i.e., $\mathcal{R}(\mathbf{A}^T)$) where $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$;

(ii) find the projection matrix $\mathbf{P}_\mathbf{A}$ such that $\mathcal{R}(\mathbf{A}) = \{\mathbf{P}_\mathbf{A}\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$, and $\mathbf{P}_{\mathbf{A}^T}$;

(iii) find the eignevalues and eigenvectors of $\mathbf{A}\mathbf{A}^T$.

(b) (10 %) Suppose that $\mathbf{A}, \mathbf{X} \in \mathbb{S}^n_+$, and $\mathrm{Tr}(\mathbf{A})$ denotes the trace of $\mathbf{A}$ (i.e., the sum of all the diagonal elements of $\mathbf{A}$).

(i) Is $\mathrm{Tr}(\mathbf{AX}) \geq 0$ true?

(ii) Is it true that if $\mathrm{Tr}(\mathbf{AX}) = 0$, then $\mathbf{AX} = \mathbf{0} \in \mathbb{R}^{n\times n}$ (zero matrix)? Prove or disprove your answer.

Problem 4. (10 %) Solve the ODEs:

(a) (5 %) the first-order ODE:

$$y' - 2xy = e^{x^2} \tag{4}$$

(b) (5 %) the second-order ODE:

$$y'' + 5y' + 4y = e^{-4x} \tag{5}$$

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Problem 5. (10 %) Find the Fourier series of $f(x)$.

(a) (5 %)

$$f(x) = x^2, -\pi \le x \le \pi, \qquad f(x + m2\pi) = f(x), m \in Z. \tag{6}$$

(b) (5 %)

$$f(x) = \{\ 1,\ \ -1 \le x \le 1\ , \qquad 0,\ x < -1;\ x > 1 \tag{7}$$

Problem 6. (5 %) Use Laplace transform to solve the following differential equation:

$$y'' + y = \delta(t - \pi), \tag{8}$$
$$y(0) = 0, \tag{9}$$
$$y'(0) = 0. \tag{10}$$

Problem 7. (10 %) Let $f(z)$ be a function of a complex variable $z = x + iy$.

(a) (5%) Assume that $Re\{f(z)\} = x^3 - 3xy^2$. Find the imaginary part $Im\{f(z)\}$ so that $f(z)$ is analytic for all $z \in \mathbb{C}$.

(b) (5%) Assume that $u(x, y) = x^3 + 3xy^2$. Show that it is impossible to find a real-valued function $v(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$ is differentiable with respect to z for all $z \in \mathbb{C}$.

Problem 8. (15 %) Let $f(z) = 1/(1 + z^3)$.

(a) (4%) Calculate $f'(0)$ and $f''(0)$.

(b) (4%) Find the Taylor series expansion of $f(z)$ about the point $z = 0$; that is, find the coefficients $\{c_n\}_{n=1}^{\infty}$ such that

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

in a certain neighborhood of $z = 0$.

(c) (2%) Continuing from part (b), what is the region of convergence?

(d) (5%) Let C denote the path along the y axis from $-\infty$ to ∞. Calculate $\int_C f(z)dz$; i.e., evaluate the following integral:

$$\int_C f(z)dz = \int_{-i\infty}^{i\infty} \frac{1}{1 + z^3}\, dz.$$