1. (5%) If square matrix $A = [a_{ij}]$ has eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ and corresponding linearly-independent eigenvectors $e_1, e_2, ..., e_n$. Which are correct? (A) If $m = n$, $A$ is diagonalizable. (B) If $A$ is diagonalizable, it is possible for $\lambda_i = \lambda_j$, $i \neq j$. (C) $a_{11} + ... + a_{nn} = \lambda_1 + ... + \lambda_n$. (D) $\{e_1, e_2, ..., e_n\}$ is always a basis of a subspace of $\mathbb{R}^n$. (E) If $e_i = a_i + i b_i$, then $a$ and $b$ are always linearly independent.

2. (5%) If $A$ is an $m \times n$ matrix, $W$ is the set of all columns of $A$, and $W^\perp$ is the orthogonal complement of $W$, then (A) $W$ is always a subspace. (B) $W^\perp$ is always a subspace. (C) $(W^\perp)^\perp = W$. (D) The intersection of $W$ and $W^\perp$ is always not an empty set. (E) dim $W^\perp + \dim (W^\perp)^\perp = n$.

3. (5%) If inconsistent linear system $Ax = b$ has a least-square solution $\hat{x}$ and $A = [a_1, a_2, ..., a_n]$, which are correct? (A) $\hat{x}$ is always existed. (B) $\hat{x}$ is always unique. (C) $A \hat{x}$ is always in $\text{span}\{a_1, a_2, ..., a_n\}$. (D) $\hat{x} = (A^T A)^{-1} A^T b$. (E) $A \hat{x} - b$ is orthogonal to rows of $A$.

4. (5%) If $A$ can be QR factorization, which are correct? (A) $A$ is a square matrix. (B) $A$ has linearly independent columns. (C) $Q^T Q = I$, where $I$ is an identity matrix. (D) $Q$ has positive entries on its diagonal. (E) $R$ is invertible.

5. (5%) If $A$ is a symmetric matrix. Which are correct? (A) $A$ is always diagonalizable. (B) $A$ has linearly independent eigenvectors. (C) $A^T A$ always has real eigenvalues. (D) The eigenvalues of $A$ are always positive. (E) $A$ can always be spectral decomposed, $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + ... + \lambda_n u_n u_n^T$.

6. (5%) Suppose that $m \times n$ matrix $A$ is invertible. Which of the following statements are true? (A) The row vectors of $A$ should be linearly independent. (B) det($A$) = 0 (C) $A$ does not have eigenvalue 0. (D) The rank of $A$ is $n$. (E) $Ax = 0$ has nontrivial solution.

7. (5%) Suppose that $A$ and $B$ are two square matrices. Determine which of the following are true. (A) det($AB$) = det($A$)det($B$). (B) det($A$) = det($A^T$). (C) det($AB$) = det($BA$). (D) det($A^{-1}$) = 1/det($A$) if $A^{-1}$ exists. (E) det($A^T$) = det($A$).

8. (5%) Determine which of the following statements are true. (A) $W = \{x \in \mathbb{R}^2 \mid x+y=1\}$ is a subspace. (B) $W = \{x \in \mathbb{R}^2 \mid x > 3y\}$ is a subspace. (C) Suppose that $W_1$ and $W_2$ are two subspaces. Then $W_1 \cap W_2$ is also a subspace. (D) Suppose that $W_1$ and $W_2$ are two subspaces. Then $W_1 \cup W_2$ is also a subspace. (E) $W = \{(a_0 + a_1 x + a_2 x^2) \mid a_0, a_1, a_2 \text{ are scalar, and } a_2 \neq 0\}$ is a subspace.

9. (5%) Suppose that $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are the standard matrices of the two transformations, $T_1$ and $T_2$, respectively. Which of the following statements are correct? (A) $T_1$ is a linear transformation. (B) $T_1$ is the transformation that counter-clockwise rotates each vector through an angle $\theta$. (C) $T_1$ is the transformation that clockwise rotates each vector through an angle $\theta$. (D) $A^{-1} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$. (E) $T_2$ is the transformation that orthogonally projects each vector onto x-axis.

10. (5%) Suppose that two $m \times n$ matrices, $A$ and $B$, are orthogonal. Which of the following statements are correct? (A) $A^T A$ is orthogonal. (B) $\det(A) = 1$ or $-1$. (C) Columns of $A$ form an orthonormal set in $\mathbb{R}^m$ with the Euclidean inner product. (D) $AB$ is an orthogonal matrix. (E) $A^T A B B^T = I$, where $I$ is an identity matrix.

11. (5%) Given a function $f$ from $A$ to $B$ and $f(a) = b$ (where $a \in A$ and $b \in B$), which of the following statements are correct? (A) $A$ is the domain of $f$. (B) $B$ is the range of $f$. (C) $f \in B^A$. (D) $b$ is the image of $a$ under $f$. (E) $a$ is the pre-image of $b$ under $f$.

12. (5%) Given the following piece of code, which of the following statements are correct?

```python
int Fibonacci(int n)
begin
    if (n == 0) or (n == 1)
        return 1;
    else
        return Fibonacci(n - 1) + Fibonacci(n - 2);
end
```

(A) This function computes Fibonacci series. (B) This is a recursive function.
16. (5%) The worst case time complexity for Euclid’s algorithm to find $\gcd(a, b)$ ($a, b$ in $\mathbb{Z}$) is (suppose $n = \max(a, b)$): (A) $\Theta(n)$. (B) $O(\log(n))$. (C) $\Theta(\log(n))$. (D) $\Theta(n \log(n))$. (E) none of the above.

17. (5%) Among the following options, which are necessary but not sufficient conditions for the corresponding goals?
   (A) “A set of total-order predicates” for “applying mathematical induction proof on those predicates”.
   (B) Assume $g, f$ are functions mapping from A to B domains, and C to D, respectively. “B, C are the same domain” for “($f \circ g$) composition is possible”.
   (C) “Existing exponential time algorithm” for “intractable (NP) problems”.
   (D) “X is a student and X is in the class” for “X is in the class only if X is a student”.
   (E) “$f$ is $O(g)$” for “$f$ is $\Theta(g)$”.

18. (5%) To analyze the complexity of the following procedure $P$, We will use the following assumptions: Suppose $P$ and $Q$ are both procedures. $Q$ take $\theta(\sqrt{m})$ time to compute, where $m$ is the size of input; each statement line in and outside the loop counts 1 step.

Procedure $P(\text{array}[a_1, a_2, \ldots, a_n])$
1. if $n<5$ exit.
2. declare initially new empty array2, array3;
3. call $Q(\text{array1})$;
4. for ($i=1$ to $n$) {
5. if (($i \mod 5$) = 1) { insert $a_i$ into array2;}
6. if ($i \mod 5$) = 3) { insert $a_i$ into array3;}
7. call $P(\text{array2})$;
8. call $P(\text{array3})$;
9. return();

Suppose $n$ is a number of power of 5, What can be the time complexity level of the procedure $P$ in the question above?
   (A) $\Theta(n)$ (B) $\Theta(\sqrt{n \log n})$ (C) $\Theta(n)$ (D) $O(n^{\log_2 3})$
   (E) $O(\sqrt{n \log n})$
19. (5%) We want to count the number of ways to climb \( n \) stairs if the climbing person can take one stair or two stairs at a time. What of the following can be the recurrence relation for our question? (initial condition: \( a_1 = 1; a_2 = 1; \))

(A) \( a_n = a_{n-1} + 1 \).

(B) \( a_n = a_{n-2} + 3 \).

(C) \( a_n = 2a_{n-2} + 1 \).

(D) \( a_n = a_{n-1} + a_{n-2} \).

(E) none of the above.

20. (5%) To solve the recurrence relation in 19., what can be the generating function \( f(z) \)?

(A) \( f(z) = 1/(1 - z - z^2) \).

(B) \( f(z) = z/(1 - z - z^2) \).

(C) \( f(z) = \frac{1}{\sqrt{5}} \left( \frac{1}{1 - (\phi + \sqrt{5}/2)z} + \frac{1}{1 - (\phi - \sqrt{5}/2)z} \right) \).

(D) \( f(z) = \frac{2}{1 - z} + \frac{2 - 3z}{(1 + 2z)^2} \).

(E) none of the above.