※請務必按照題號次序寫在答案紙上，否則將嚴重失分。

1.(10%) Suppose x represents students, y represents courses. Let $G(y)$ be the predicate of "y is a graduate level class", $U(y)$ be the predicate of "y is a under-graduate level course", $F(x)$ be the predicate of "x is a freshman student", $M(x)$ be the predicate of "x is a master student", $T(x,y)$ be the predicate of "x has taken y". Choose the correct sentences that the following logic statements represent:

a) (5%) $\forall x, y, G(y) \wedge F(x) \rightarrow \neg T(x, y)$

(*i*) Only freshman students have not taken graduate level courses.
(*ii*) No freshman has taken graduate level courses.
(*iii*) Not every graduate level course has been taken by every freshman student.
(*iv*) Some freshman have not taken graduate level courses.

b) (5%) $\forall x, y_2, \exists y_1, (G(y_1) \wedge U(y_2) \wedge T(x, y_1) \wedge (\neg T(x, y_2))) \rightarrow M(x)$

(*i*) Some master students have taken graduate level courses without taken every under-graduate level courses.
(*ii*) Every master students have taken all graduate level courses but not have taken every under-graduate level courses.
(*iii*) Some master students have taken all graduate level courses without having taken any under-graduate level courses.
(*iv*) Only master students have taken some graduate level courses without having taken any under-graduate level courses.

(注意：上述 *a*), *b*) 各小題全對才給5分)

2.(5%) Suppose every element in A is different from others in A, and $|A| = 9$, $a \in A$. Find the cardinality of the following set, (no points will be awarded if only partially correct)

$\{B \mid ((B \subseteq A) \wedge (|B| = 3) \wedge (a \in B))\}$, $|B| = ?$

3.(10%) A researcher uses the following mathematic induction procedure to prove the statement that "all the observed animals are the same species" is true.

First, obviously each single animal is its own species.

Assuming all groups of 1 to (n-1) animals are the same species, we can induct that any group of n animals must belong to the same species because of the following argument:

Any group of n can be partitioned two ways --- the first one and the rest of (n-1), and the first (n-1) and the last one. Since all (n-1) or smaller groups are the same species, the middle overlapping (n-2) group must belongs to the same species as the first and the last one. So all the n animals are the same species.

Discuss why this prove is correct or incorrect.

4.(25%) A graph G consists of a set V of vertices (or nodes) and a set E of edges (or arcs) such that each edge e is associated with an unordered pair of vertices. Let $V = \{1, 2, 3, \ldots n\}$

(*a*) (6%) How many graphs are there with vertex set V?
(*b*) (6%) How many of the graphs in (*a*) contain the triangle 123 ?
(*c*) (6%) What is the total number of triangles in all the graphs with vertex V ?
(*d*) (7%) On average, how many triangles does a graph on n labeled vertices contain ?

(還有第二頁)

※請務必按照題號次序寫在答案紙上，否則將嚴重失分。

5.(8%) Let A be a $m \times n$ matrix. Suppose there are k free variables in $A\mathbf{x} = \mathbf{0}$. Which of the following are true. (im A = Col A, ker A = Null A)

(*a*) Dim (ker A^T)= $m-n+k$

(*b*) Rank A = Rank A^T

(*c*) Det (AA^T)= Det (A^TA)

(*d*) im (A)= im (AA^T).

6.(7%) Express the image of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 5 \end{bmatrix}$ as the kernel of a matrix $\boldsymbol{B}$.

7.(10%) Let $\boldsymbol{B}$ be the basis of $\boldsymbol{R}^n$ consisting of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and let I be some other basis of $\boldsymbol{R}^n$. Is $[\vec{v}_1]_I, [\vec{v}_2]_I, \ldots, [\vec{v}_n]_I$ a basis of $\boldsymbol{R}^n$ as well? Explain.

8.(16%) True and false (每小題答對給2分，答錯扣2分，不答0分，本題總分≥0)

(*a*) A plane in $\boldsymbol{R}^3$ is a two-dimensional subspace of $\boldsymbol{R}^3$.

(*b*) $\{(a, c, b-1, a-c): a, b, c \text{ in } R\}$ is a subspace.

(*c*) $\{(a, b): a\,b \geq 0\}$ is a subspace.

(*d*) $\{(a, b, c): a - 3b + c = 0, b - 2c = 0, 2b - c = 0\}$ is a subspace.

(*e*) Matrix A is invertible if and only if A has no zero eigenvalue.

(*f*) If $A\mathbf{x} = \lambda\,\mathbf{x}$ for some vector $\mathbf{x}$, then λ may be not an eigenvalue of A.

(*g*) A row replacement operation on matrix A does not change the eigenvalues of A.

(*h*) If $n \times n$ matrix A has n linearly independent eigenvectors, then A can be factorized into QR.

9.(9%) Suppose $A = PDP^{-1}$, where D is a diagonal square matrix. If $\mathcal{B}$ is the basis for $\boldsymbol{R}^n$ formed from the columns of P, show that D is the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

(後面沒有題目了)