國立中央大學九十一學年度碩士班研究生入學試題卷

1. (10%) Find the interval of convergence of the series $\sum_{n=1}^{\infty} s_k x^k$ where s_k is the kth partial sum of the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

2. (15%)
   (1) Find $\lim_{n\to\infty} (\sqrt{n}-1)^{1/\sqrt{n}}$.
   (2) If $f(x) = \int_2^{\sqrt{x}} \frac{dt}{\sqrt{1+t^4}}$, find $(f^{-1})'(0)$.
   (3) Compute $\int x\sqrt{6x - x^2 - 8}\, dx$.

3. (20%)
   (1) Find the values of p for which the series $\sum_{k=2}^{\infty} \frac{1}{k(lnk)^p}$ converges.
   (2) Let $p > 1$. Use the integral test to show that

$$\frac{1}{(p-1)(n+1)^{p-1}} < \sum_{k=1}^{\infty} \frac{1}{k^p} - \sum_{k=1}^{n} \frac{1}{k^p} < \frac{1}{(p-1)n^{p-1}}.$$

4. (15%) Let $I_n = \int_0^{\infty} \frac{x^{2n-1}}{(x^2+1)^{n+3}} dx$, $n \geq 1$. Prove that $I_n = \frac{n-1}{n+2} I_{n-1}$ and evaluate $\int_0^{\infty} \frac{x^5}{(x^2+1)^6} dx$.

5. (10%) Set

$$g(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

   (a) Show that $\partial g/\partial x$ and $\partial g/\partial y$ both exist at (0,0). What are their values at (0,0)?
   (b) Is the limit $\lim_{(x,y)\to(0,0)} g(x,y)$ exist or not? Why?

6. (20%)
   (1) Use cylindrical coordinates to compute

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{\sqrt{9-x^2-y^2}} \frac{1}{\sqrt{x^2+y^2}} dz dx dy.$$

   (2) Compute $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2} dz dx dy$.

7. (10%) Use triple integration to find the volume of the solid T bounded above by the parabolic cylinder $z = 4 - y^2$ and bounded below by the elliptic paraboloid $z = x^2 + 3y^2$.

參考用