Instructions: Answer the following questions. Make and state your own assumptions for questions where the information is not sufficient for you to solve them. For example, if you need the corresponding p-value of a normally distributed random variable evaluated at 2.5, you may indicate the value as, say, $Pr(x \geq 2.5)$, where $x \sim \mathcal{N}(0,1)$.

1. (40 %) Let $U(a,b)$ denote a uniform distribution whose values range between a and b. A random variable (r.v.) y is defined as follows:

$$y = sx_1 + (1-s)x_2$$

where the r.v. s has a Bernoulli distribution, which takes the value of 1 with probability $\frac{1}{3}$; x_1 and x_2 are two random variables having uniform distributions (and $x_1 \sim U(-2,1)$ and $x_2 \sim (0,3)$). s, x_1 and x_2 are independent. That is, there is a probability of $\frac{1}{3}$ that the r.v. is generated from $U(-2,1)$ and a probability of $\frac{2}{3}$ that the r.v. is from

   (a) Calculate the mean and variance of y.

   (b) Given that $y = 0.5$, what is the probability that y is generated from $U(-2,1)$.

   (c) Calculate the probability that y is greater than $\frac{1}{2}$, i.e., $Pr(y > \frac{1}{2})$.

   (d) If $y = \frac{1}{3}x_1 + \frac{2}{3}x_2$, what is the probability that y is greater than $\frac{1}{2}$, i.e., $Pr(y > \frac{1}{2})$?

2. (10%) If (x_1, x_2) are a random sample from a Bernoulli distribution, , which takes the value of 1 with probability p. Suppose you are asked to test $H_0 : p = \frac{1}{2}$ against the alternative hypothesis $H_0 : p \neq \frac{1}{2}$, and you decide to reject the null hypothesis whenever $|\frac{1}{2}(x_1 + x_2) - \frac{1}{2}| \geq \frac{1}{8}$. What is your type I error?

3. (25%) In a certain population the random variable Y has variance equal to 360. Two independent random samples, each of size 20, are drawn. The first sample mean is used as the predictor of the second sample mean.

   (a) (15%) Calculate the expectation, expected square, and variance, of the prediction error.

   (b) (10%) Approximate the probability that the prediction error is less than 12 in absolute value.

4. (25%) The random variable X has the power distribution on the interval [0, 1]. That is, the pdf of X is

$$f(x;\theta) = \theta x^{\theta-1} \quad \text{for } 0 \leq x \leq 1,$$

with $f(x;\theta) = 0$ elsewhere. The parameter θ is unknown. Consider random sampling, sample size n.

   (a) (15%) Show that the maximum likelihood estimator of θ is $T = 1/\overline{Y}$, where $Y = -\log X$. ("log" denotes natural logarithm.)

   (b) (10 %) Find the asymptotic distribution of T, in terms of θ and n only.