# 國立中央大學101學年度碩士班考試入學試題卷

所別：數學系碩士班 甲組(一般生)　　科目：高等微積分　　共 1 頁 第 1 頁
　　　數學系碩士班 甲組(在職生)

本科考試禁用計算器　　　　＊請在試卷答案卷（卡）內作答

1. Determine (by proof or counterexample) the truth or falsity of the following statements :

(a) If $\{x_n\}$ and $\{y_n\}$ are bounded sequences of real numbers, then $(\liminf x_n)(\liminf y_n) \le \liminf(x_n y_n)$ . (10%)

(b) Let $(M,d)$ be a metric space. For each fixed $x \in M$ and $\varepsilon > 0$, the set $\{y \in M : d(x,y) < \varepsilon\}$ is open. (10%)

(c) Let $(M,d)$ be a metric space and let $\{x_n\}_{n=1}^{\infty} \subseteq M$ converge to $x \in M$. Then $A = \{x_1, x_2, \cdots\} \cup \{x\}$ is compact. (10%)

(d) Every continuous real-valued function on [0,2] is Riemann integrable. (10%)

(e) The function $f(x) = x^2$ is uniformly continuous on $[0,\infty)$. (10%)

(f) Let $\{f_k\}_{k=1}^{\infty}$ be the sequence of functions on $[0,1]$ defined by

$$f_k(x) = \begin{cases} 4k^2 x & \text{if } 0 \le x < \frac{1}{2k} \\ 4k - 4k^2 x & \text{if } \frac{1}{2k} \le x < \frac{1}{k} \\ 0 & \text{if } \frac{1}{k} \le x \le 1. \end{cases}$$

Then $\{f_k\}$ converges uniformly on $[0,1]$. (10%)

2. Let $(M,d)$ be a compact metric space and $\phi : M \mapsto M$ satisfy $d(\phi(x),\phi(y)) < d(x,y)$ for all $x,y \in M, x \ne y$. Show that $\phi$ has a unique point $\xi \in M$ such that $\phi(\xi) = \xi$. (20%)

3. Let $\mathbb{R}$ be the real number system and $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Suppose $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \ne (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Investigate the differentiability of $f$ at $(0,0)$. (20%)