(20%) 1. Let \( A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \\ 1 & 4 & 2 & 3 \\ 1 & 3 & 3 & 3 \end{pmatrix} \) and \( b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \) and \( c = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

(a) Find the reduced row echelon form of \( A \) and the rank of \( A \). (5%)
(b) Find the inverse of \( A \) if it exists. (5%)
(c) Find the set of solutions of the linear system \( AX = b \). (5%)
(d) Find the set of solutions of the linear system \( AX = c \). (5%)

(30%) 2. Let \( A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \).

(a) Determine the characteristic polynomial of \( A \) and find the eigenvalues of \( A \). (6%)
(b) For each eigenvalue \( \lambda \) of \( A \), find the eigenspace corresponding to \( \lambda \). (8%)
(c) Determine whether \( A \) is diagonalizable. Explain why. (4%)
(d) Determine the Jordan canonical form of \( A \). (4%)
(e) Compute \( A^{101} \). (8%)

(20%) 3. Let \( V \) be the space of functions from \( \mathbb{R} \) to \( \mathbb{R} \) and let \( U \) be the subset consisting of continuous functions in \( V \). Let \( S = \{1, \sin x, \cos x\} \) and \( W = \text{Span}(S) \). Define \( T : W \to W \) by \( T(f) = f' \), where \( f' \) is the derivative of \( f \).

(a) Prove or disprove that \( U \) is a subspace of \( V \). (4%)
(b) Show that \( S \) is a basis for \( W \). (4%)
(c) Show that \( T \) is a linear transformation and find the null space of \( T \). (4%)
(d) Find the matrix representation of \( T \) in the ordered basis \( S \). (4%)
(e) For each eigenvalue \( \lambda \) of \( A \), find the set of eigenvectors corresponding to \( \lambda \). (4%)

(15%) 4. (a) State the Dimension Theorem. (5%)
(b) Let \( A, B \in M_{n \times n}(\mathbb{R}) \). Prove or disprove that \( \text{rank}(A) \geq \text{rank}(AB) \). (5%)
(c) Let \( A, B \in M_{n \times n}(\mathbb{R}) \). Prove or disprove that \( \text{rank}(B) \geq \text{rank}(AB) \). (5%)

(15%) 5. Let \( V = \mathbb{R}^4 \), \( S = \{(0,0,1,1),(0,1,0,1),(0,1,1,0)\} \) and \( W = \text{Span}(S) \).

(a) Find an orthogonal basis for \( W \). (8%)
(b) Determine the dimension of the orthogonal complement of \( W \). (3%)
(c) Find an orthogonal basis for the orthogonal complement of \( W \). (4%)