國立中央大學 107 學年度碩士班考試入學試題

所別： 數學系 碩士班 數學組(一般生)　　　　　　　　　共 1 頁　第 1 頁
　　　　數學系 碩士班 應用數學組(一般生)
　　　　數學系 碩士班 應用數學組(在職生)

科目： 線性代數

本科考試禁用計算器　　　　　　　　　　　　　　　＊請在答案卷(卡)內作答

Instructions: Do all problems. Show your work. Notations: V is a vector space over $\mathbb{R}$ or $\mathbb{C}$, $P_n(\mathbb{R})$ denotes the set of all polynomials with real valued coefficients and degree at most n.

1. Let $A = \begin{pmatrix} -1 & 3 & -1 \\ -1 & 2 & 0 \\ 1 & -3 & 3 \end{pmatrix}$.

   (a) Find the minimal polynomial of A. (10%)
   (b) Explain why A is diagonalizable or not diagonalizable. (5%)
   (c) Find a Jordan canonical form J of A. (5%)
   (d) Find a matrix Q such that $Q^{-1}AQ = J$. (5%)

2. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the map defined by
$$T(f(x)) = f''(x) + f'(x) - f(x).$$

   (a) Show that T is a linear map. (6%)
   (b) Find the matrix $[T]_\beta$ representing T with respect to the standard ordered basis $\beta = \{1, x, x^2\}$ of $P_2(\mathbb{R})$. (6%)
   (c) Find the inverse matrix of $[T]_\beta$, if it exists. (7%)
   (d) Find T^{-1}, if it exists. (6%)

3. Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be the linear operator defined by $T(z_1, z_2, z_3) = (z_1, 2z_2, -z_3)$. Express T^{-1} as a polynomial of T. Prove your answer. (10%)

4. A linear operator $P : V \to V$ is called a projection if $P^2 = P$. Prove that if P is a projection, then $V = \text{null}\,P \oplus \text{range}\,P$. (10%)

5. Let $V = P_n(\mathbb{R})$ with the inner product $< f, g >= \int_{-1}^{1} x^2 f(x)g(x)dx$, and consider the subspace $P_3(\mathbb{R})$ with the standard ordered basis $\beta = \{1, x, x^2, x^3\}$. Use the Gram-Schmidt process to replace β by an orthogonal basis $\{v_1, v_2, v_3, v_4\}$ for $P_3(\mathbb{R})$. (10%)

6. Suppose V is finite-dimensional and $T : V \to V$ is a linear operator. Prove that $V = \text{null}\,T + \text{range}\,T$ if and only if $\text{null}\,T \cap \text{range}\,T = \{0\}$. (10%)

7. Let $U = \{p \in P_4(\mathbb{R}) : p'(2) = 0\}$. Find a basis of U. (10%)

參考用