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科目： 高等微積分
本科考試禁用計算器

Write legibly and logically. Decide how much details to include.

Part I: Basics: Complete the sentence in the case of a definition

1. (7ps) Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$. We say that f is uniformly continuous on A if ...

2. (7ps) A set sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is Cauchy if ...

3. (7ps) Let $S \subset \mathbb{R}$ be a bounded set. A real number $\alpha = sup\, S$, the supremum of the set S if α satisfies ....

4. (7ps) Let $A \subset \mathbb{R}^n$, the boundary of the set A denoted by ∂A is the set ...

5. (7ps) State the Generalized (Cauchy's) Mean Value Theorem for differentiable functions.

6. (7ps) State the Fundamental Theorem of Calculus.

7. (7ps) State the Intermediate Value Theorem.

Part II: Computations

8. (10ps) If $P(x) = x^{27} + 3x^{26} - x^2 + 23$, Find $\lim_{x\to\infty} P(x)^{\frac{1}{27}} - x$.

9. (10ps) Determine if the function $f(x) = \sum_{n=1}^{\infty} \left(\frac{x^n}{n!}\right)^2$ is continuous on $\mathbb{R}$ or not.

10. (11ps) Find the maximum and minimum values of $f(x, y, z) = y^2 - 10z$ subject to the constraint: $x^2 + y^2 + z^2 = 36$.

Part III: Proofs

11a). (7ps) Let $f_k : [a, b] \to \mathbb{R}$ be a sequence of (Riemann) integrable functions. Suppose the f_k's converge uniformly to f on $[a, b]$. Prove that f is also (Riemann) integrable on $[a, b]$ and $\lim_{k\to\infty} \int_a^b f_k(x)\,dx = \int_a^b f(x)\,dx$.

11b). (3ps) Give an example showing (and write the proof for it) that part a) is not true if the f_k are just converging pointwise to f.

12. (10ps) True or false? If you think the following statement is false, give a counter-example (and prove that your example works) and if you think that it is true, prove it. Let $X \subset \mathbb{R}^n$ be an open set and $f : X \to \mathbb{R}$, $x_o \in X$, $\vec{n} \in \mathbb{R}^n$ is a unit vector. Suppose f is differentiable in every direction $\vec{n}$ at x_o, then f is differentiable at x_o.