國立中央大學 112 學年度碩士班考試入學試題

所別： 數學系碩士班

科目： 高等微積分

共 5 題，每題 20 分。第 1～4 題與第 5 題的 (a) 都是證明題。第 5 題的 (b) 需要計算過程，無計算過程者不予與計分。

$\mathbb{R}$ 表示實數的集合，$\mathbb{Z}$ 表示整數的集合。

1. Let $f : \mathbb{R}^2 \to \mathbb{Z}$ be a continuous function and

$$S = \{(x, y) \in \mathbb{R}^2 : y = 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = 0\}.$$

Assume $f(0,0) = 0$. Prove or disprove that $f(x, y) = 0$ for all $(x, y) \in S$.

2. Let (M, d) be a metric space and $f : M \to M$ satisfy

$$d\big(f(x), f(y)\big) < d(x, y) \qquad \text{for all } x, y \in M \text{ with } x \neq y.$$

Prove that if M is compact, then there is a unique point $p \in M$ such that $f(p) = p$.

3. The power series $\sum_{n=0}^{\infty} c_n x^n$ converges at $x = R$, where the coefficients c_n are fixed real numbers. Given $0 < \varepsilon < R$, show that $\sum_{n=0}^{\infty} c_n x^n$ converges uniformly on $[-R + \varepsilon, R - \varepsilon]$.

4. Define

$$f(x) = \left(\int_0^x e^{-t^2} \, dt \right)^2, \quad g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1} \, dt.$$

Show that $f(x) + g(x) = \dfrac{\pi}{4}$ for all $x \in \mathbb{R}$.

5. Let $\begin{cases} u(x, y) = x^3 - \dfrac{y^4}{x} \\ v(x, y) = \sin x - \cos y \end{cases}$.

   (a) Show that the map $(x, y) \mapsto (u, v)$ is locally invertible at $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.

   (b) Compute $\dfrac{\partial x}{\partial v}$ at $(x, y) = \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.