

國立中央大學 112 學年度碩士班考試入學試題

所別： 數學系碩士班

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科目： 微積分

總共十題，每題十分，均為計算證明題，請給出計算與證明細節(否則不予計分)。

(1) Consider the function

$$f(x) = \begin{cases} x^5 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

- (a) (4 %) For  $x \neq 0$ , find  $f'(x)$ .  
 (b) (3 %) Find  $f'(0)$ .  
 (c) (3 %) Show that  $f'(x)$  is continuous for all real numbers.

- (2) (a) (5 %) Find  $\frac{d^2y}{dx^2}$  if  $x^2 \cos^2 y - \sin y = 0$ .  
 (b) (5 %) Consider the closed curve given by the polar equation  $r = 2 - \sin \theta$ ,  $0 \leq \theta \leq 2\pi$ . Find the area of the region enclosed by the curve.

- (3) (a) (5 %) Find  $\lim_{x \rightarrow 1^+} (\ln x)^{x-1}$ .  
 (b) (5 %) Determine if the improper integral  $\int_0^\infty \frac{1}{\sqrt{x^3+1}} dx$  converges or diverges.

(4) (10 %) Evaluate

$$\lim_{n \rightarrow \infty} \left( \frac{1}{\sqrt{n}\sqrt{2n+1}} + \frac{1}{\sqrt{n}\sqrt{2n+2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{2n+n}} \right).$$

- (5) (10 %) Let  $f: [0, 1] \rightarrow \mathbb{R}$  be a continuous function such that  $\int_0^1 f(x) dx = \frac{1}{2}$ . Show that there exists a point  $c \in [0, 1]$  such that  $f(c) = c$ .

(6) (10 %) Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{n3^n}.$$

(7) Let

$$f(x, y) = \begin{cases} \frac{3xy}{x^2-y^2} & \text{if } x^2 - y^2 \neq 0 \\ 0 & \text{if } x^2 - y^2 = 0. \end{cases}$$

- (a) (4 %) Use the definition of partial derivative to find  $\frac{\partial f}{\partial y}$  at  $(x, y) = (0, 0)$ .  
 (b) (6 %) Show that  $f$  is not differentiable at  $(0, 0)$ .

- (8) (10 %) Find all relative extrema and saddle points of the function  $f(x, y) = x^3 + y^3 - 3xy$ .

- (9) (10 %) Find the volume of the solid (in  $\mathbb{R}^3$ ) between two spheres:  $x^2 + y^2 + z^2 = 1$ ;  $x^2 + y^2 + z^2 = 4$ , and inside the cone  $z^2 = x^2 + y^2$ .

- (10) (10 %) Evaluate the double integral  $\int_0^3 \int_0^{3-x} (y-2x)^2 \sqrt{x+y} dy dx$ .