國立中央大學八十六學年度碩士班研究生入學試題卷

所別: 數學研究所 不分組 科目: 線性代數

1. Let A be an $m \times n$ matrix with $m < n$. Show that $\det(A^T A) = 0$, where A^T is the transpose of A. (15%)
2. Find the rank for each of the following matrices: (15%)

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 2 & 2 \\ 0 & 3 & 3 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 3 & 0 & 3 & 3 \\ -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

3. Find the inverses of the following matrices if they exist. (20%)

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 2 & 2 & -1 \\ 3 & 1 & 3 & 2 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

4. Let $A = \begin{bmatrix} 9 & -2 & -1 \\ -3 & 10 & -1 \\ -3 & -2 & 11 \end{bmatrix}$. Find a diagonal matrix D and an invertible matrix C such that $C^{-1}AC = D$. (15%)
5. Let A be a symmetric $n \times n$ matrix. Let u and v be eigenvectors corresponding to two distinct eigenvalues of A. Show that u and v are orthogonal. (15%)
6. Let $W = \{[x, y, z, w] : x + y - 2z - 3w = 0 \text{ and } x - 2y - z + 2w = 0\}$. Let $b = [1, 1, 1, -1]$. Find v in W such that the distance from v to b is the shortest one. (10%)
7. Show that a linear transformation maps a linear subspace to a linear subspace. (10%)