國立中央大學九十學年度碩士班研究生入學試題卷

所別: 數學系 不分組 科目: 數值分析 共 1 頁 第 1 頁

1. (**10 points**: interpolation)
   Construct a divided-difference table to find the Hermite polynomial $H_5(x)$ for the function $f(x)$ using the following data:

i	x_i	$f(x_i)$	$f'(x_i)$
0	1.3	0.620	-0.522
1	1.6	0.455	-0.570
2	1.9	0.282	-0.581

2. (**10 points**: least squares approximation)
   Find the least squares approximating polynomial of degree two, $P_2(x)$, for the function $f(x) = \sin(\pi x)$ on the interval $[0,1]$.
   (Note that $\int_0^1 \sin(\pi x)dx = \frac{2}{\pi}$, $\int_0^1 x\sin(\pi x)dx = \frac{1}{\pi}$, $\int_0^1 x^2 \sin(\pi x)dx = \frac{\pi^2-4}{\pi^3}$)

3. (**10 points**: Gaussian quadrature)
   Determine constants a, b, c, and d that will produce a quadrature formula

$$\int_{-1}^{1} f(x)dx = af(-1) + bf(1) + cf'(-1) + df'(1)$$

   that gives exact results for polynomials of degree 3 or less.

4. (**10 points**: composite numerical integration)
   Suppose that $f(0.25) = f(0.75) = \alpha$. Find α if the composite trapezoidal rule with $n = 2$ gives the value 4 for $\int_0^1 f(x)dx$, and with $n = 4$ gives the value $\frac{7}{2}$.

5. (**20 points**: numerical differentiation)
   Assume that $f \in C^4[x_0 - h, x_0 + h]$. Using the Taylor's theorem to derive the following three-point formula for approximating $f''(x_0)$:

$$f''(x_0) = \frac{1}{h^2}\Big\{f(x_0 - h) - 2f(x_0) + f(x_0 + h)\Big\} - \frac{h^2}{12}f^{(4)}(\xi),$$

   for some $\xi \in (x_0 - h, x_0 + h)$.

6. (**20 points**: root-finding problem)
   (1). Derive directly the formula of Newton's method for solving the nonlinear system

$$\begin{cases} f(x,y) &= 0, \\ g(x,y) &= 0. \end{cases}$$

   (2). Starting with $(1,1)$, perform two iterations of Newton's method on the following system:

$$\begin{cases} xy^2 + x^2y + x^4 &= 3, \\ x^3y^5 - 2x^5y - x^2 &= -2. \end{cases}$$

7. (**20 points**: iterative technique for solving $A\mathbf{x} = \mathbf{b}$)
   Let $A = [a_{ij}] \in \mathbf{R}^{n\times n}$, $a_{ii} \neq 0$, for $i = 1, \cdots, n$, and $\mathbf{b} = (b_1, \cdots, b_n)^t \in \mathbf{R}^n$. The basic concept of iterative methods for solving the system of linear equations $A\mathbf{x} = \mathbf{b}$ is to convert the system into an equivalent system of the form $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ and lead to the iterations

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}.$$

   (1). Find T and $\mathbf{c}$ for the Gauss-Seidel method.
   (2). Please give a necessary and sufficient condition to ensure the convergence of the Gauss-Seidel method for any initial guess $\mathbf{x}^{(0)} \in \mathbf{R}^n$.