以下各題，只給答案，沒有說明，不給分

1. (a) (8 分) Let N, H be two groups. Assume that there exits a group homomorphism $\phi : H \to \mathrm{Aut}(N)$, where $\mathrm{Aut}(N)$ is the group of automorphisms of N, then the semi-direct product $N \rtimes H$ of N and H (with respect to ϕ) is the set $N \times H$ together with the binary operation $*$ such that $(n_1, h_1) * (n_2, h_2) = (n_1\, \phi(h_1)(n_2), h_1 h_2)$ for every $n_1, n_2 \in N$ and $h_1, h_2 \in H$. Show that $N \rtimes H$ is a group under the binary operation $*$.

   (b) (7 分) Let G be a group and let N, H be subgroups of G. Assume that N is normal in G. For $h \in H$, let $i_h(n) = hnh^{-1}, \forall n \in N$. Verify that the map $\phi(h) = i_h$ is a group homomorphism from H to $\mathrm{Aut}(N)$. Assume that $N \cap H = \{e\}$ and $G = NH$. Show that G is isomorphic to the semi-direct product of N and H (with respect to ϕ).

   (c) (10 分) Let $N, H \leq G$ be as in (b) such that N is normal in G. Assume that $N \cap H = \{e\}$ and $G = NH$. Let J be a subgroup of G. Consider the set $H_J = \{h \in H \mid nh \in J \text{ for some } n \in N\}$. Show that H_J is a subgroup of H and is isomorphic to the quotient group $J/J \cap N$.

2. (a) (10 分) Let S_5 denote the symmetric group of degree 5 (i.e. the permutation group of 5 letters). What is the order of a Sylow 5-subgroup of S_5? How many Sylow 5-subgroups does S_5 have? You need to explain your answers.

   (b) (10 分) Let G be a finite group of order n and let m be a divisor of n. Assume that there are exactly r ($r \geq 1$) subgroups of G which are of order m. Let H be any subgroup of G of order m and let $N(H)$ denote the normalizer of H. Show that if $r \nmid n$ then $[G : N(H)] < r$.

3. (10 分) Let M be an Abelian group. A homomorphism of M into itself is called an endomorphism of M. Let $\mathrm{End}(M)$ be the set of all endomorphisms of M. Define multiplication on $\mathrm{End}(M)$ by function composition and addition on $\mathrm{End}(M)$ by $(\phi + \psi)(m) = \phi(m) + \psi(m), \forall m \in M$ and $\phi, \psi \in \mathrm{End}(M)$. It is already known that $\mathrm{End}(M)$ forms a ring under the multiplication and addition defined above. Let M_1, M_2 be two Abelian groups. Prove or disprove that $\mathrm{End}(M_1 \times M_2)$ is isomorphic to $\mathrm{End}(M_1) \times \mathrm{End}(M_2)$.

4. (a) (10 分) Show that in a principal ideal domain, a non-zero ideal is prime if and only if it is a maximal ideal.

(b) (15 分) Let $\mathbb{Q}[x]$ denote the polynomial ring with coefficients in $\mathbb{Q}$. For any $f(x) \in \mathbb{Q}[x]$, define $\phi(f) = f(\sqrt{2} - \sqrt{3})$. Verify that $\phi : \mathbb{Q}[x] \to \mathbb{R}$ is a ring homomorphism, where $\mathbb{R}$ is considered as a ring under the usual addition and multiplication of real numbers. Show that the image of ϕ is a subfield of the field of real numbers $\mathbb{R}$ and describe what this subfield is.

5. Recall that a finite field is a field consisting of finitely many elements.

(a) (8 分) Let $\mathbb{F}$ be a finite field. Prove that the number of elements of $\mathbb{F}$ is equal to p^f for some prime number p and some integer $f \geq 1$.

(b) (12 分) Construct a finite field of 25 elements.