參考用

科目：線性代數(1002)　　校系所組：中大數學系甲組、乙組　交大應用數學系乙組
清大數學系純粹數學組、應用數學組

考試時間 100 分鐘, 分數共 100 分.

All vector spaces and matrices in this set of problems are over the real field $\mathbb{R}$.

一、複選題 A (每小題全對才給分, 無部份分數)

1. [5%] Let V be the vector space of real valued functions on the closed interval $[0,1]$. Then in the following, pick up the linearly independent sets.
   (a) The elements are rational functions $\frac{1}{x+1}, \frac{1}{x+2}, \frac{1}{x+3}, \ldots$.
   (b) The elements are rational functions $\frac{1}{(x+1)}, \frac{1}{(x+1)^2}, \frac{1}{(x+1)^3}, \ldots$.
   (c) The elements are polynomial functions $1+x$, $1+x+x^2$, $1+x+x^2+x^3$, ...
   (d) The elements are polynomials $(x+1), (x+2)^2, (x+3)^3, (x+4)^4, \ldots$.
   (e) The elements of trigonometric functions $\cos x$, $\cos 2x$, $\cos^2 x$, $\cos 3x$, $\cos^3 x$, ...

2. [5%] Let A, B denote two $n \times n$ matrices satisfying $AB = 0$. Then in the following, pick up the correct statements.
   (a) $BA = 0$,
   (b) all eigenvalues of BA are 0,
   (c) $(BA)^2 = 0$,
   (d) $A = 0$ or $B = 0$,
   (e) rank A+rank $B = n$.

3. [5%] Let A be an $n \times n$ matrix with entries over $\mathbb{R}$ such that $A^2 = -I_n$, where I_n is the $n \times n$ identity matrix. Pick up the correct statements.
   (a) A can not be a symmetric matrix.
   (b) n must be even.
   (c) The rank of A is not zero.
   (d) The trace of A is not zero.
   (e) If B is another $n \times n$ matrix with entries over $\mathbb{R}$ such that $B^2 = -I_n$, then A and B are similar.

二、複選題 B (每小題全對才給分, 無部份分數)

1. [5%] If $T : V \to W$ is a linear map on the vector spaces V and W, then in the following, pick up the correct statements.
   (a) If V_0 is a subspace of V, then $T(V_0) = \{T(x) \mid x \in V_0\}$ is also a subspace of W.
   (b) If W_0 is a subspace of W, then $T^{-1}(W_0) = \{x \mid T(x) \in W_0\}$ is also a subspace of V.
   (c) If T is one-to-one, then T is onto.

注意：背面有試題

(d) If $U : V \to W$ is another linear map, and T and U agree on a basis for V, then $T = U$.

2. [5%] Both A and B are $n \times n$ matrices. Then in the following, pick up the correct statements

(a) If A and B are similar, then $\det(A) = \det(B)$.

(b) If A has the rank n, then $\det(A) = 0$.

(c) If $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is the $2n \times 2n$ matrix, then $\det(M) = \det(A)\det(B)$, where 0 is the zero matrix.

(d) If B is obtained from A by performing an elementary row operation, then there exists an elementary matrix E such that $B = AE$.

3. [5%] Let V be a vector space with dimension n. Then in the following, pick up the correct statements.

(a) Any linearly independent subset for V containing exactly n vectors is a basis for V.

(b) Any finite generating set for V contains at most n vectors.

(c) Any two bases for V have the same number of vectors.

(d) If $\{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ is a basis for V, then

$$\{v_1, v_1 + 2v_2, v_1 + 2v_2 + 3v_3, v_1 + 2v_2 + 3v_3 + 4v_4, \ldots, v_1 + 2v_2 + 3v_3 + \cdots + nv_n\}$$

is also a basis for V.

三、填充題 (只看結果評分, 無部份分數)

1. [5%] Let $\theta < \eta$ denote two eigenvalues of $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$, with $\begin{pmatrix} \theta \\ a \end{pmatrix}$ the eigenvector corresponding to θ and $\begin{pmatrix} \eta \\ b \end{pmatrix}$ the eigenvector corresponding to η. Let $A^{100} = \begin{pmatrix} 1 & c \\ 0 & 2^{100} \end{pmatrix}$. What is $a + b + c$?

2. [5%] Determine the minimal polynomial of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}.$$

3. [5%] Determine the minimal polynomial of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}^{-1}.$$

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4. [5%] Determine the characteristic polynomial of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

5. [5%] Let V be the set of polynomials of degree 5 or less in x over $\mathbb{R}$ and $D : V \to V$ is defined by $D(p(x)) = \frac{d}{dx}(p(x))$. Find the trace of D.

四、計算證明題 (依解答的完整性評分)

1. [9%] Find a 3×3 matrix A such that

$$2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx = \begin{pmatrix} x & y & z \end{pmatrix} AA^t \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where A^t is the transpose of A.

2. [9%] Let A be an $n \times m$ matrix of rank r such that every set of r rows and every set of r columns is linearly independent. Prove that every $r \times r$ submatrix of A is nonsingular.

3. [9%] Let A be an $n \times n$ matrix, and we have

$$A^3 + 3A^2 - A = 3I_n,$$

where I_n is the $n \times n$ identity matrix. Then prove that A is diagonalizable.

4. [9%] Let $T : V \to W$ and $U : W \to Z$ be linear maps on the finite dimensional vector spaces V, W and Z. Then prove

$$\begin{aligned} \text{Rank}(UT) &\leq \text{Rank}(U), \\ \text{Rank}(UT) &\leq \text{Rank}(T), \end{aligned}$$

where UT is the composite map of U and T, and $\text{Rank}(f)$ is the rank of the map f.

5. [9%] Let $T : V \to W$ be a surjective linear map on the vector spaces V and W, let $N(T)$ be the null space of T, and suppose V, W have the bases α, β respectively . Then prove

$$N(T) \oplus W \cong V,$$

where the vector space $N(T) \oplus W = \{(x, y) \mid x \in N(T), y \in W\}$ has the operations

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2), \\ c(x, y) &= (cx, cy), c \in \mathbb{R}. \end{aligned}$$

Note: V and W are not necessarily finite dimensional vector spaces.