1. (Total 10%) A particle moving in x-y plane whose potential energy is
\[ U = \frac{1}{2} k (x^2 + y^2) \] and \( k \) is a constant
(i) (5%) What is the force acting on the particle?
(ii) (5%) Prove that the angular momentum is conserved for this particle.

2. (Total 30%) A binary system consists of two stars of masses \( m_1 \) and \( m_2 \) respectively orbiting around their center of mass in circular orbits. Suppose the separation of two stars is \( a \). Using the center of mass as the origin of the coordinate system,
(i) (10%) Show that the Kepler’s 3rd law of this binary system can be written as
\[ \frac{a^3}{P^2} = \frac{G(m_1 + m_2)}{4\pi^2} \]
where \( P \) is the orbital period of the binary system and \( G \) is the gravitational constant.
(ii) (5%) What is the total energy \( E \) of the system? Write it down as a function of \( m_1 , m_2 \), and \( a \).
(iii) (15%) The total energy of the binary system would be lost due to emission of gravitational wave with a rate of
\[ \frac{dE}{dt} = -\frac{32G^4 (m_1 + m_2)m_1^2 m_2^2}{5c^5 a^5} \]
Suppose the energy loss rate is very small so the orbit is still circular and the Kepler’s 3rd law is still hold during the evolution. What is \( \frac{dP}{dt} \)? Write it down as a function of \( m_1 , m_2 \) and \( P \).

3. (Total 20%) There is a system of the monoatomic ideal gas with fixed molecular number \( N \) in a container
(i) (5%) Please derive the entropy \( S \) of as the function of its internal energy \( U \) and volume \( V \), that is \( S(U, V) \)
(ii) (5%) What is the final temperature \( T_2 \) for the system processing adiabatic expansion (絶熱膨脹) from \( V_1 \) to \( V_2 \) with the initial temperature of \( T_1 \)
(iii) (5%) What is the entropy change for the system processing isothermal expansion
(Total 5%) What is the entropy change for the system processing isobar expansion (等壓膨脹) from $V_1$ to $V_2$.

4. (Total 15%) There is a sphere of radius $R$ with spherical symmetry charge density $\rho(r) = \rho_c \left(1 - \frac{r}{R}\right)$ where $\rho_c$ is the charge density at the center.

(i) (5%) What is the total charge in this sphere?

(ii) (10%) Derive the electric fields and potential for $r < R$ and $r > R$ if we define $\Phi(r \to \infty) = 0$.

5. (10%) An antiproton can be made by colliding two protons as $p + p \to p + p + p + \bar{p}$ where $p$ is proton and $\bar{p}$ is antiproton. Suppose initially one of the protons (proton 1) is rest, what is the minimum kinetic for another proton (proton 2) to make this reaction happen if the rest mass of proton is $M_p$.

6. (Total 15%) In quantum mechanics, the parity operator $\hat{P}$ is the flip in the sign of spatial coordinate, that is, $x \to -x$ in one dimensional system. For example, for any function $\varphi(x)$, $\hat{P}\varphi(x) = \varphi(-x)$

(i) (5%) Show that the only possible eigenvalues of $\hat{P}$ is either 1 or -1.

(ii) (10%) Suppose $u_h(x)$ is the eigenfunction of Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$ with eigenenergy of $E$ where $\hat{p}$ is momentum operator, show that $\hat{P}u_h(x)$ is also an eigenfunction of $\hat{H}$ with same eigenenergy if the potential $V(x)$ is an even function.