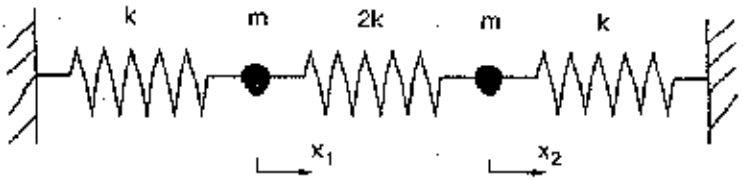
1. Consider two identical particles with mass m connected by three springs. Assume that th particles can only move in the horizontal direction.



(a) (5 pts) Write down the equations of motion for x_1 and x_2.

(b) (10 pts) Solve the equations of motion to obtain the two eigenmodes and their frequencies

2. (a) (7 pts) Solve the operator equation $(D^8 + 8D^6 + 16D^4)y(x) = 0$ where $D^n = \frac{d^n}{dx^n}$.

(b) (13 pts) Solve the following differential equation

$$\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = 0 \quad \text{with} \quad V(x) = \begin{cases} 1 & x < 0 \\ -1 & x > 0 \end{cases}$$

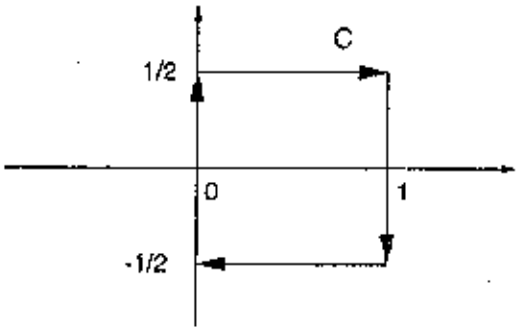
and boundary conditions

$$\psi(x \to \infty) = 0, \qquad \psi(x = 0) = 1.$$

3. Given the integral on the complex plane,

$$I = \int_C \frac{dz}{z - 1}$$

with the contour C defined as in the following figure



(a) (10 pts) Compute the integrations of the four segments of the contour C separately. Add the four results together to get I.

(b) (10 pts) Use the Residue Theorem to compute the integration I. Check the result with that obtained in (a).

4. (10 pts) A gas of total N atoms each with mass m is in equilibrium at a temperature T and under a potential field $V(x, y, z)$,

$$V(x, y, z) = -(x^2 + y^2 + z^2).$$

Boltzmann told us that the number density $P(x, y, z)$ at a given location is proportional to $e^{-mV/kT}$.

(a) Find the number density $P(x, y, z)$ with the correct normalization constant.

(b) Define the radial coordinate $r^2 = x^2 + y^2 + z^2$ such that

$$dxdydz = 4\pi r^2 dr.$$

Find $P(r)$.

5. (15 pts) Find the Fourier cosine series for the function

$$f(x) = \begin{cases} 1 & 0 < x < \frac{L}{2} \\ 0 & \frac{L}{2} < x < L \end{cases}$$

6. (20 pts) Solve the Laplace equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

in the region $0 < x < 1$ and $0 < y < 1$ with the following boundary conditions

$$\begin{aligned} \psi(x = 0, y) &= 0 \\ \psi(x = 1, y) &= 0 \\ \psi(x, y = 0) &= 0 \\ \psi(x, y = 1) &= \sin 2\pi x. \end{aligned}$$