

類組：物理類 科目：應用數學(2001)

※請在答案卷內作答

1. (a) Write down the Cauchy integral formula and prove it. (15 points)

(b) Apply (a) to calculate $\oint_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz$ (10 points)

(c) If $f(z)$ is analytical inside and on a closed contour C , calculate

$$\oint_C \frac{f'(z)}{f(z)} dz \quad (10 \text{ points})$$

2. (a) A square matrix A is "orthogonally diagonalizable" if there exists an orthogonal matrix C such that $D = C^T A C$ is a diagonal matrix. Prove that A is orthogonally diagonalizable if and only if it is a symmetric matrix. (10 points)

(b) Orthogonally diagonalize $A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$ and what are the matrices C and D ? (10 points)

3. Find the Fourier transform of the function

$$f(t) = \begin{cases} 0, & t < 0 \\ e^{-t/\tau} \sin \omega_0 t, & t \geq 0 \end{cases} \quad (10 \text{ points})$$

4. The vector field \mathbf{F} is given by

$$\mathbf{F} = (3x^2yz + y^3z + xe^{-x})\mathbf{i} + (3xy^2z + x^3z + ye^x)\mathbf{j} + (x^3y + y^3x + xy^2z^2)\mathbf{k}$$

Calculate the value of the line integral $\int_L \mathbf{F} \cdot d\mathbf{r}$ where L is the 3D closed contour OABCDEO defined by the successive vertices $(0, 0, 0), (1, 0, 0), (1, 0, 1), (1, 1, 1), (1, 1, 0), (0, 1, 0), (0, 0, 0)$. (10 points)

5. Solve

$$\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

subject to the boundary condition $u(0, y) = 0$ and $u(x, 1) = x^2$. (10 points)

6. The Laplace transform of $f(t)$ is defined as

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

Show that

(a) $\mathcal{L}\left[\frac{d^2 f(t)}{dt^2}\right] = -\frac{df}{dt}\Big|_0 + s(sF(s) - f(0))$

(b) $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}$, for $n = 1, 2, 3, \dots$

(15 points)