皆為計算題，請詳列計算過程，無計算過程者不予計分
左側方格內數字為各小題配分

1. [*20 points*] **Second-Order Linear ODE**
Consider the differential equation

$$\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + b^2 x = 0\,,$$

where $x(t)$ is a real function, and both a and b are positive constants. Find the general solutions of this differential equation for the following three cases:

5 (a) $a > b$

5 (b) $a = b$

10 (c) $a < b$. Use the constant $w = \sqrt{b^2 - a^2}$ to express the derivation and solution for this case wherever applicable.

2. [*20 points*] **Vector Analysis**

(a) Given a vector field $\vec{H} = 2xy\hat{x} + (x^2 + z^2)\hat{y} + 2yz\hat{z}$. (Here $\hat{x}$, $\hat{y}$ and $\hat{z}$ are unit vectors in the directions of the x, y, and z axes, respectively).

5 i. Find the divergence of $\vec{H}$.

7 ii. Evaluate the flux of $\vec{H}$ over the surface bounded by the rectangular region defined by $0 \le x \le 1$, $1 \le y \le 2$, $-1 \le z \le 3$.

8 (b) Given a vector function $\vec{E} = y\hat{x} + x\hat{y}$. Evaluate the integral $\int_C \vec{E} \cdot \vec{dl}$ from $(3, 0, 0)$ to $\frac{3}{\sqrt{2}}(1, 1, 0)$ along the circle of radius $r = 3$.

3. [*15 points*] **Matrix Diagonalization**
Consider the matrix

$$A = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}.$$

5 (a) Find the eigenvalues of A.

5 (b) Find the normalized eigenvectors of A.

5 (c) Work out the matrix Q such that $Q^{-1}AQ$ is diagonal.

4. [*25 points*] **Gaussian Integrals**

5 (a) Show that

$$\left(\int_{-\infty}^{\infty} dx\, e^{-ax^2}\right)^2 = \frac{\pi}{a}$$

10 (b) Evaluate the integral

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-(ax^2 + 2bxy + cy^2)}\,,$$

expressed in terms of the eigenvalues λ and ξ of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

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10 (c) As a specific case in the extension of the analysis in (b), evaluate the integral

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N \, e^{-\sum_{j=1}^{N}(2x_j^2 - 2x_j x_{j+1})},$$

for $N \gg 1$, where we denote $x_{N+1} \equiv x_1$.
Note that the integrand of the integral corresponds to the Boltzmann weight for the potential energy of an N-bead chain in one dimension subject to the periodic boundary condition, where each bead is connected by a spring with each of its two neighbors. Here, we assume N is so large that $\frac{1}{(2N+1)} \sum_{j=-N}^{N} \exp(i\frac{(k-l)2\pi}{N} j) \approx \delta_{kl}$ is a good approximation, for integers $1 \le k, l \le N$.

5. [*20 points*] **One-Dimensional Poisson Equation**

10 (a) Solve the boundary value problem

$$\frac{d^2 G(x,x')}{dx^2} = -\delta(x - x') \quad \text{on the interval } (0,1),$$
$$G(x=0,x') = G(x=1,x') = 0,$$

where $\delta(x-x')$ is the Dirac delta function at a point $x = x'$, between 0 and 1 ($0 < x' < 1$). The solution is the Green's function for the one-dimensional Poisson equation satisfying the given boundary conditions.

10 (b) Solve the boundary value problem

$$\frac{d^2\phi}{dx^2} = -x^2 \quad \text{on } (0,1),$$
$$\phi(0) = \phi(1) = 0,$$

using the Green's function obtained in (a).

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