Notation: In the following questions, underlined letters such as $a, b,$ etc. denote column vectors of proper length; boldface letters such as $A, B,$ etc. denote matrices of proper size; $A^T$ means the transpose of matrix $A$. $I_n$ is the $(n \times n)$ identity matrix. $\|a\|$ means the Euclidean norm of vector $a$. $\mathbb{R}$ is the usual set of all real numbers. $\det(A)$ is the determinant of square matrix $A$. row$(A)$ and col$(A)$ are the row and column spaces of $A$ over $\mathbb{R}$, respectively. For any linear map $T$ over vector spaces, we use $\ker(T)$, $\text{rank}(T)$ and $\text{nullity}(T)$ for the kernel, rank and nullity of $T$, respectively. Let $W$ be a subspace of $\mathbb{R}^n$; then by $W^\perp$ we mean the orthogonal complement of $W$ in the Euclidean inner product space $\mathbb{R}^n$. $\mathcal{L}: f(t) \mapsto F(s)$ and $\mathcal{L}^{-1}: F(s) \mapsto f(t)$ denote the unilateral Laplace and inverse Laplace transforms for $t \geq 0$, respectively.

一、 Which of the following sets is basis (are bases) for $\mathbb{R}^3$?

(A) $\{[0,1,0]^T, [0,0,1]^T, [1,0,0]^T\}$.

(B) $\{-2,4,-6\}^T, [1,-2,3]^T\}$.

(C) $\{[0.14,0,-0.1]^T, [-1,-0.2,0.4]^T, [0.5,0.5,-1]^T\}.$


(E) None of the above are true.
二、 Which of the following statements about the multiplicative inverse of a matrix is/are true?

(A) A matrix $A$ is called invertible if there exists a matrix $B$ such that $AB$ is an identity matrix.

(B) If a matrix is both diagonalizable and invertible, then so is its multiplicative inverse.

(C) Suppose that matrices $A$ of size $n \times n$ and $D$ of size $m \times m$ are invertible and that matrix $C$ is of size $m \times n$; then the following identity is true

\[
\begin{bmatrix}
A & 0 \\
C & D
\end{bmatrix}^{-1} =
\begin{bmatrix}
A^{-1} & 0 \\
-D^{-1}CA^{-1} & D^{-1}
\end{bmatrix}.
\]

(D) Methods for finding the multiplicative inverse of a matrix include LU factorization, Gaussian elimination, eigen-decomposition, and Gram-Schmidt process.

(E) None of the above are true.

三、 Which of the following properties of eigenvalue is/are true?

(A) A scalar $\lambda$ is an eigenvalue of matrix $A$ if and only if $\lambda$ is an eigenvalue of $A^\top$.

(B) A matrix is positive semi-definite if and only if all of its eigenvalues are non-negative.

(C) Every eigenvalue of a matrix $A$ is also an eigenvalue of $A^2$.

(D) If matrices $A$ and $B$ are similar, then they have the same eigenvalues.

(E) None of the above are true.
四、Which of the following matrices is/are diagonalizable?

(A) \[
\begin{bmatrix}
3 & -1 \\
0 & 3 \\
\end{bmatrix}
\]

(B) \[
\begin{bmatrix}
5 & 0 \\
1 & 5 \\
0 & 0 & 4 \\
\end{bmatrix}
\]

(C) \[
\begin{bmatrix}
-2 & 8 & -4 \\
-6 & 8 & 0 \\
-6 & 2 & 6 \\
\end{bmatrix}
\]

(D) \[
\begin{bmatrix}
2 & 0 & -2 & 9 \\
0 & 2 & 1 & 0 \\
0 & 0 & 3 & -3 \\
0 & 0 & 0 & 5 \\
\end{bmatrix}
\]

(E) None of the above are true.

五、Which of the following statements about matrix factorization is/are true?

(A) If a matrix \( A \) is positive definite, then \( A \) has “an” LU factorization, \( A = LU \), where the diagonal entries of \( U \) are positive.

(B) Suppose that a matrix \( A = QR \), where \( Q \) is an \( m \times n \) matrix and \( R \) is an \( n \times n \) matrix. If the columns of \( A \) are linearly independent, then \( R \) must be invertible.

(C) Any factorization of a matrix \( A = UDV^T \), with matrices \( U, V \) square and positive diagonal entries in the matrix \( D \), is called a singular value decomposition of \( A \).

(D) An \( n \times n \) matrix \( A \) is positive definite if and only if \( A \) has “a” Cholesky factorization \( A = R^T R \) for some invertible upper triangular matrix \( R \) whose diagonal entries are all positive.

(E) None of the above are true.
六、 The Householder matrix $H = I_n - 2\text{proj}_u$, where $\text{proj}_u = \frac{1}{\|u\|^2} uu^T$ is the orthogonal projection matrix onto some nonzero vector $u \in \mathbb{R}^n$, is a reflection matrix. Which of the following statements is/are true?

(A) Consider the 2-dimensional space, i.e. $n = 2$. For vectors $u, d, p, q \in \mathbb{R}^2$ shown in the figure below, we have $Hd = p$.

(B) $H$ is a symmetric and orthogonal matrix.

(C) Both linear systems $A\underline{x} = b$ and $HA\underline{x} = Hb$ are equivalent.

(D) Let $a = [a_1, \ldots, a_n]^T$ and $u = [a_1 - \|a\|, a_2, \ldots, a_n]^T \neq 0$. Then $\|u\|^2 = -2\|a\|u_1$ and $Ha = [\|a\|, 0, \ldots, 0]^T$.

(E) None of the above are true.
七、 Given

\[
A = \begin{bmatrix}
1 & 11 & 23 & 81 & 97 \\
2 & 22 & 46 & 162 & 194 \\
3 & 1 & 11 & 2 & 1 \\
9 & 0 & 1 & -4 & 3 \\
2 & 5 & 3 & 2 & 1
\end{bmatrix}, \quad
B = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
7 & 6 & 0 & 0 & 0 \\
3 & 9 & 2 & 0 & 0 \\
3 & -4 & 0 & -1 & 0 \\
1 & 2 & 101 & -5 & -2
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
7 & 1 & 0 & 5 & 37 \\
3 & 20 & 0 & 9 & 71 \\
8 & -71 & 0 & 1 & 13 \\
2 & -2 & 0 & 2 & 3 \\
1 & -5 & 0 & 45 & 1
\end{bmatrix}, \quad
D = \begin{bmatrix}
0 & 1 & 0 & 4 & 0 & -2 \\
37 & 20 & 7 & 2 & 1 & 4 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & -10 & -1 & 0 & -2 \\
2 & -5 & -94 & 0 & 0 & 4 \\
0 & -1 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
E = \begin{bmatrix}
7 & 1 & 23 & 0 & 0 & 1 \\
1 & 2 & 101 & -5 & -2 & 0 \\
3 & -4 & 0 & -1 & 0 & 0 \\
3 & 9 & 2 & 0 & 0 & 0 \\
7 & 6 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad
F = \begin{bmatrix}
1 & 0 & 0 & 2 & 9 \\
7 & 2 & 0 & 3 & 7 \\
3 & 7 & 1 & 5 & -9 \\
0 & 0 & 0 & -4 & 3 \\
0 & 0 & 0 & 2 & 1
\end{bmatrix}
\]

which of the following statements is/are true?

(A) \( \det(A) = \det(C) = 0. \)

(B) \( \det(D) = -40. \)

(C) \( \det(B) = \det(E). \)

(D) \( \det(F) = 20. \)

(E) None of the above are true.
八、Consider a $3 \times 3$ nonzero matrix $A$, and let $W = \text{col}(A)$ and $V = \text{col}(A^T)$. The least squares solutions $\varepsilon_{LS}$ to $Ax = b$ are illustrated in the following figure, where $\text{proj}_W b$ is the orthogonal projection of vector $b$ onto vector space $W$. A minimum length least squares solution $\varepsilon_{MILLS}$ is the one among the least squares solutions that has a minimum norm.

Which of the following statements is/are true?

(A) $(\text{col}(A))^\perp = \ker(A^T)$ is proved by either (1) if $x \in (\text{col}(A))^\perp$, then $x^T Ax = 0$ for any $x$; and then $A^T x = 0$; or (2) if $y \in \ker(A^T)$, then $y^T Ax = 0$ for any $x$.

(B) Let $h = b - \text{proj}_W b$ (see the above figure). Then $A^T h = 0$ and the system $Ax = h + y$ for any $y \in \text{col}(A)$ is a consistent system, i.e., $Ax = h + y$ has at least one solution (not least squares solution).

(C) A least squares solution $\varepsilon_{LS}$ to $Ax = b$ satisfies $Ax = \text{proj}_W b$, and then satisfies $A^T Ax = A^T b$.

(D) The minimum length least squares solution $\varepsilon_{MILLS}$ to $Ax = b$ is unique, and $\varepsilon_{MILLS} = \text{proj}_V \varepsilon_{LS}$, where $V = \text{col}(A^T)$.

(E) None of the above are true.
九．Let \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) be the eigenvectors of matrix \( \mathbf{A} \) corresponding respectively to eigenvalues \( \lambda_1 \) and \( \lambda_2 \), where
\[
\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}.
\]
It is found that \( \lambda_1 > 1 > |\lambda_2| \). With matrix \( \mathbf{A} \), let \( \mathbf{x}_n = [x_{n,1}, x_{n,2}]^\top \in \mathbb{R}^2 \) for \( n = 0, 1, \ldots \) be a series of vectors related by \( \mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n \). Given the initial condition \( \mathbf{x}_0 = [1, 0]^\top = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 \) for some \( \alpha, \beta \in \mathbb{R} \), which of the following statements is/are true?

(A) \( \beta = \frac{1}{\lambda_1 - \lambda_2} \).

(B) \( \mathbf{x}_n = \alpha(\lambda_1)^n \mathbf{v}_1 + \beta(\lambda_2)^n \mathbf{v}_2 \).

(C) \( \lim_{n \to \infty} \frac{x_{n,1}}{x_{n+1,1}} = \lambda_1 \).

(D) \( \lim_{n \to \infty} \frac{x_{n,2}}{x_{n+1,2}} = \lambda_1 \).

(E) None of the above are true.
Let $T_u$ be a linear transformation on $\mathbb{R}^3$ for a rotation by an angle $\theta$ about a unit vector $u$. Specifically, we let the matrix for $T_u$ with respect to the standard basis $S$ for $\mathbb{R}^3$ be

$$\mathbf{G} = [T_u]_S = \begin{bmatrix} c + u_1^2(1-c) & u_1u_2(1-c) - u_3s & u_1u_3(1-c) + u_2s \\ u_1u_2(1-c) + u_3s & c + u_2^2(1-c) & u_2u_3(1-c) - u_1s \\ u_1u_3(1-c) - u_2s & u_2u_3(1-c) + u_1s & c + u_3^2(1-c) \end{bmatrix},$$

where $[u]_S = [u_1, u_2, u_3]^T$ is the coordinate vector of $u$ with respect to $S$, $c = \cos(\theta)$ and $s = \sin(\theta)$. Furthermore, let $\mathbf{A}$ be the rotation matrix about the $z$-axis of Cartesian coordinate system by an angle $\theta$, i.e.,

$$\mathbf{A} = [T_\theta]_S = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $[u]_S = [0, 0, 1]^T$. Let $B = \{\mathbf{n}, \mathbf{b}, \mathbf{u}\}$ be an ordered orthonormal basis for $\mathbb{R}^3$ with $[\mathbf{n}]_S = [n_1, n_2, n_3]^T$ and $[\mathbf{b}]_S = [b_1, b_2, b_3]^T$ and let $\mathbf{P}_{S \leftarrow B} = [[n]_S, [b]_S, [u]_S]$ be the change-of-basis matrix for changing basis from $B$ to $S$. Which of the following statements is/are true?

(A) $n_1^2 + b_1^2 + u_1^2 = 1$, $n_1n_2 + b_1b_2 + u_1u_2 = 0$ and $n_1n_3 + b_1b_3 + u_1u_3 = 0$.

(B) The coordinate vector of $\mathbf{u}$ with respect to basis $B$ is $[\mathbf{u}]_B = \mathbf{P}_{S \leftarrow B} [\mathbf{u}]_S$.

(C) $\mathbf{G} = \mathbf{P}_{S \leftarrow B} \mathbf{A}$.

(D) $\mathbf{G} = \mathbf{A} \mathbf{P}_{S \leftarrow B}$.

(E) The matrix for $T_u$ with respect to basis $B$ is $\mathbf{A}$.
十一、 Solve for $y(x)$ the first order differential equation

$$xy'(x) - 4x^2y(x) + 2y(x)\ln(y(x)) = 0$$

by the substitution $v = \ln(y(x))$. Which of the following statements is/are true?

(A) It is a nonlinear ordinary differential equation for the dependent variable $y$.

(B) It is a nonlinear ordinary differential equation for the new variable $v$.

(C) There exists a solution $y(x)$ satisfying the condition $y(0) = 1$.

(D) There exists a solution $y(x)$ satisfying the condition $y(1) = 1$.

(E) None of the above are true.
十二、 The Cauchy-Euler equation

\[ x^2y''(x) - 4xy'(x) + 6y(x) = 0 \]

can be transformed into a constant coefficient equation \( y''(v) + by'(v) + cy(v) = 0 \) by the substitution \( v = \ln(x) \). Which of the following statements is/are true?

(A) \( b = -5 \).

(B) \( c = -6 \).

(C) The solution \( y(x) \) exists only for \( x > 0 \).

(D) \( y(x) = C_1x^2 + C_2x^3 \) for some constants \( C_1 \) and \( C_2 \).

(E) None of the above are true.

十三、 Continued from Problem 十二. Solve the non-homogeneous second order differential equation

\[ x^2y''(x) - 4xy'(x) + 6y(x) = x^3 \]

by variation of parameters, i.e., set the particular solution as \( y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \), where \( y_1(x) \) and \( y_2(x) \) are homogeneous solutions. With initial conditions \( y(1) = 0 \) and \( y'(1) = 1 \) for the complete solution \( y(x) \), which of the following statements is/are true?

(A) The real valued solution \( y(x) \) exists only for \( x > 0 \).

(B) \( y(2) = 2 \).
The first order system

\[
\begin{bmatrix}
    x'_1(t) \\
    x'_2(t) \\
    x'_3(t) \\
    x'_4(t)
\end{bmatrix} =
\begin{bmatrix}
    0 & 1 & 0 & 0 \\
    8 & 0 & 17 & 0 \\
    0 & 0 & 0 & 1 \\
    17 & 0 & 8 & 0
\end{bmatrix}
\begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    x_3(t) \\
    x_4(t)
\end{bmatrix}
\]

can be reduced into an equivalent second order system \( y''(t) = By(t) \) with \( y(t) = [x_1(t), x_3(t)]^T \). Which of the following statements is/are true?

(A) \( B = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} \).

(B) \( B = \begin{bmatrix} 8 & 17 \\ 17 & 8 \end{bmatrix} \).

(C) The eigenvalues of \( B \) are 9 and 25.

(D) \([1, 1]^T\) is an eigenvector for \( B \).

(E) None of the above are true.
十五、 Continued from Problem 十四. Find the particular solution for the second order system 
\[ y''(t) = By(t) \] with initial conditions \( y(0) = [1, 3]^T \) and \( y'(0) = [-3, 3]^T \). Which of the following statements is/are true?

(A) \( x_1(t) = e^{-5t} + e^{5t} - \cos(3t) - \sin(3t) \).
(B) \( x_3(t) = e^{-5t} + e^{5t} + e^{3t} \).
(C) \( (x_1(t) + x_3(t)) \) is an odd function in \( t \).
(D) \( (x_1(t) - x_3(t)) \) is an odd function in \( t \).
(E) None of the above are true.

十六、 For \( s > 0 \), let \( F(s) \) be the unilateral Laplace transform of function \( f(t) \) given by

\[ F(s) = \frac{1}{2s^2} - \frac{1}{s(e^s + e^{3s})}. \]

Which of the following statements regarding the values of \( f(t) \) is/are true?

(A) \( f(1) = \frac{1}{2} \).
(B) \( f(2) = 1 \).
(C) \( f(4) = 2 \).
(D) \( f(8) = 3 \).
(E) None of the above are true.
十七. Consider the following periodic function

\[ f(t) = \sum_{n=-\infty}^{\infty} \exp(-\pi(t-n)^2) \]

which has a Fourier series representation

\[ f(t) = \sum_{m \geq 0} a_m \cos(2\pi mt) + b_m \sin(2\pi mt) \]

for some \( a_m, b_m \in \mathbb{R} \) and for all \( t \in \mathbb{R} \). Which of the following statements is/are true?

(A) \( a_0 = \frac{1}{\sqrt{\pi}} \).
(B) \( a_1 = e^{-\pi} \).
(C) \( a_2 = \frac{3}{\pi^2} \).
(D) \( b_2 = e^{-4\pi} \).
(E) None of the above are true.
十八. For the following second order differential equation

\[ tx''(t) + (4t - 2)x'(t) + (2t - 4)x(t) = 0 \]

Let \( x_1(t) = t^{r_1} \sum_{n \geq 0} a_n t^n \) and \( x_2(t) = t^{r_2} \sum_{n \geq 0} b_n t^n \) be the two linearly independent Frobenius series solutions for \( x(t) \) when \( t > 0 \), where \( r_1 \) and \( r_2 \) are the zeros of the corresponding indicial equation. Assume \( r_1 \geq r_2 \) and \( a_0 = b_0 = 1 \). Which of the following statements is/are true?

(A) \( r_1 - r_2 \) is not an integer.
(B) \( a_2 = \frac{13}{3} \).
(C) \( a_3 = -\frac{6}{18} \).
(D) \( b_3 = \frac{4}{3} \).
(E) None of the above are true.
十九、Continued from Problem 十八. The second order differential equation can be alternatively solved by using Laplace transform. Assuming \( x(0) = 0 \) and \( \int_0^\infty x(t)dt = 1 \), which of the following statements is/are true about the values of \( x(t) \) and its unilateral Laplace transform \( X(s) = \mathcal{L}\{x(t)\} \)?

(A) \( x'(0) = 1 \).
(B) \( x(1) = 1 \).
(C) \( X(1) < 1 \).
(D) Values of \( X(s) \) exists for all \( s > -1 \).
(E) None of the above are true.
Consider the following boundary value problem for the bivariate function \( u(x, t) \) defined for \( 0 \leq x \leq \pi \) and \( t \geq 0 \)

\[
\frac{\partial}{\partial t} u(x, t) = 2 \frac{\partial^2}{\partial x^2} u(x, t) + u(x, t)
\]

Given the end-point and initial conditions

\[
\left. \frac{\partial}{\partial x} u(x, t) \right|_{x=0} = \left. \frac{\partial}{\partial x} u(x, t) \right|_{x=\pi} = 0 \quad \text{and} \quad u(x, 0) = x(\pi - x)
\]

which of the following statements is/are true for the solution \( u(x, t) \) when it is expressed as

\[
u(x, t) = \sum_{n \geq 0} a_n e^{\imath n t} \cos(2nx) + b_n e^{\imath n t} \sin(2nx)
\]

for some constants \( a_n, b_n, p_n, q_n \in \mathbb{R} \)?

(A) \( a_0 = \frac{\pi}{6} \).

(B) \( a_1 = -1 \).

(C) \( p_2 = -31 \).

(D) \( b_1 = -1 \).

(E) None of the above are true.