- 本測驗試題為多選題（答案可能有一個或多個），請選出所有正確或最適當的答案，並請將答案用2B鉛筆填於答案卡。
- 共二十題，每題五分。每題ABCDE選項單獨計分；每一選項個別分數為一分，答錯倒扣一分，倒扣至本測驗試題零分為止。

Notation: In the following problems, $\mathbb{R}$ is the usual set of all real numbers. We will use underlined letters such as $\underline{a} \in \mathbb{R}^n$ to denote a real, column vector $\underline{a}$ of length n. $\|\underline{a}\|$ means the Frobenius norm of vector $\underline{a}$, and $\underline{0}$ is the all-zero column vector of proper length. We will use boldface letters such as $\mathbf{A} \in \mathbb{R}^{m\times n}$ to denote a real matrix $\mathbf{A}$ of size $m \times n$, and we will write $\mathbf{A} = [a_{i,j}] \in \mathbb{R}^{m\times n}$, where $a_{i,j}$ is the (i,j)-th entry of $\mathbf{A}$ with subindices $i = 1,\ldots,m$, and $j = 1,\ldots,n$. $\mathbf{A}^\top$ is the transpose of matrix $\mathbf{A}$. rank($\mathbf{A}$) denotes the rank of matrix $\mathbf{A}$. $\mathbf{I}_n$ is the $n \times n$ identity matrix. det($\mathbf{A}$) and tr($\mathbf{A}$) are respectively the determinant and trace of square matrix $\mathbf{A}$. row($\mathbf{A}$), col($\mathbf{A}$) and null(A) are the row, column and right null spaces of $\mathbf{A}$ over $\mathbb{R}$, respectively. Unless otherwise stated, all vector spaces and linear combinations are over field $\mathbb{R}$, and the orthogonality is with respect to the usual Euclidean inner product. By dim($\mathcal{W}$) we mean the dimension of vector space $\mathcal{W}$ over its base field $\mathbb{R}$. $\mathscr{L} : f(t) \mapsto F(s)$ and $\mathscr{L}^{-1} : F(s) \mapsto f(t)$ denote the unilateral Laplace and inverse Laplace transforms for $t \geq 0$, respectively. Primes of functions of one variable denote the derivatives with respect to the variable, for instance, $y'(x) = \frac{d}{dx}y(x)$, $y''(x) = \frac{d^2}{dx^2}y(x)$, etc.

1. Consider the matrix:

$$\begin{bmatrix} 3 & 1 & 2 & 4 & 5 \\ 2 & 1 & 3 & 4 & 6 \\ 4 & 7 & 8 & -1 & -2 \end{bmatrix}$$

which has reduced row echelon form of the following

$$\begin{bmatrix} 1 & 0 & 0 & x & y \\ 0 & 1 & 0 & z & w \\ 0 & 0 & 1 & u & \frac{54}{23} \end{bmatrix}.$$

Which of the following statements is/are true?

(A) $x = \frac{29}{23}$.

(B) $y = -\frac{31}{23}$

(C) $z = \frac{51}{23}$.

(D) $w = -\frac{86}{23}$.

(E) None of the above is true.

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2. Let

$$A = \frac{1}{8}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$$

and

$$(A^{-1})^{10} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

Which of the following statements is/are true?

(A) $x = 524288$.

(B) $y = 524288$.

(C) $z = 1048576$.

(D) $w = 1048576$.

(E) None of the above is true.

3. Given the matrix A below

$$A = \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & -1 & -1 & 3 \\ 1 & 4 & -1 & -2 \end{bmatrix},$$

which of the following statements is/are true?

(A) $\dim(\text{col}(A)) = 3$.

(B) $\dim(\text{row}(A)) = 2$.

(C) $\dim(\text{null}(A)) = 1$.

(D) $\dim(\text{null}(A^\top)) = 1$

(E) None of the above is true.

4. Suppose $\mathcal{T}:\mathbb{R}^3\to\mathbb{R}^2$ is a linear transformation such that

$$\mathcal{T}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}4\\8\end{bmatrix},\ \mathcal{T}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\6\end{bmatrix},\ \mathcal{T}\left(\begin{bmatrix}2\\1\\1\end{bmatrix}\right)=\begin{bmatrix}5\\10\end{bmatrix}.$$

Now if for any $[x_1,x_2,x_3]^\top\in\mathbb{R}^3$

$$\mathcal{T}\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}a\\b\end{bmatrix},$$

then which of the following statements is/are true?

(A) $a+b=3x_1+3x_2+6x_3$.

(B) $a-2b=-3x_1+3x_2-6x_3$.

(C) $2a+b=4x_1+4x_2+2x_3$.

(D) $b-a=x_1+x_2+2x_3$.

(E) None of the above is true.

5. Continue from Problem 4. The following is a basis for the null space (kernel) of linear transformation $\mathcal{T}$

$$\left\{\begin{bmatrix}-1\\1\\a\end{bmatrix},\begin{bmatrix}-2\\0\\b\end{bmatrix}\right\}.$$

Which of the following statements is/are true?

(A) $a+b=1$.

(B) $a-b=0$.

(C) $a/b=2$.

(D) $a\sqrt{b}=0$.

(E) None of the above is true.

6. Consider the linear system $\mathrm{A}\underline{x}=\underline{b}$ in the unknown $\underline{x}$, where $\mathrm{A}\in\mathbb{R}^{n\times n}$ and $\underline{b}\in\mathbb{R}^n$ are both nonzero. Which of the following statements is/are true?

(A) The system is consistent if $\underline{b}^\top\mathrm{A}\neq\underline{0}^\top$.

(B) The system can be consistent if $\underline{b}^\top\mathrm{A}=\underline{0}^\top$.

(C) The system is inconsistent only if A has nullity larger than zero.

(D) The system has infinitely many solutions only if A has a zero eigenvalue.

(E) None of the above is true.

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7. For the square matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \end{bmatrix},$$

which of the following statements is/are true?

(A) $\text{rank}(\mathbf{A}) = 3$.

(B) $\det(\mathbf{A}\mathbf{A}^\top) = -2$.

(C) The linear system $\mathbf{A}^\top\mathbf{A}\underline{x} = \underline{b}$ in the unknown $\underline{x}$ is consistent for every $\underline{b} \in \mathbb{R}^3$.

(D) The matrix $\mathbf{A}^\top\mathbf{A}$ has an eigenvector $\underline{e}$ with $\|\underline{e}\| = 1$ such that $\mathbf{A}\underline{e} = \underline{0}$.

(E) None of the above is true.

8. A symmetric positive-definite matrix $\mathbf{A} = [a_{i,j}] \in \mathbb{R}^{3\times 3}$ has eigenvalues 1, 1, 3. Which of the following statements is/are true?

(A) The maximum of the quadratic form $\underline{x}^\top\mathbf{A}\underline{x}$ over all $\underline{x} \in \mathbb{R}^3$ subject to $\|\underline{x}\| = 1$ is 6.

(B) The product $a_{1,1}a_{2,2}a_{3,3}$ of diagonal entries can be negative.

(C) The matrix $\mathbf{A} - \mathbf{I}_3$ has nullity equal to one.

(D) Any matrix $\mathbf{B} \in \mathbb{R}^{3\times 3}$ with eigenvalues equal to 1, 1, 3 must be similar to $\mathbf{A}$.

(E) None of the above is true.

9. For

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \text{ and } \underline{b} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix},$$

which of the following statements is/are true?

(A) The orthogonal projection of $\underline{b}$ onto $\text{col}(\mathbf{A})$ is $[3, 3, 3]^\top$.

(B) $\min_{\underline{x}\in\mathbb{R}^2} \|\mathbf{A}\underline{x} - \underline{b}\|^2 = 6$.

(C) Let $\mathcal{L}$ be the set of least squares solutions to the system $\mathbf{A}\underline{x} = \underline{b}$ in the unknown $\underline{x}$; then $\mathcal{L}$ is a subspace of $\mathbb{R}^2$.

(D) $\min_{\underline{x}\in\mathcal{L}} \|\underline{x}\|^2 = \frac{1}{4}$, where $\mathcal{L}$ is defined in (C).

(E) None of the above is true.

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10. Consider the transformation $\mathcal{T}: \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ given by $\mathcal{T}(\underline{x}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \underline{x}^\top \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Which of the following statements is/are true?

(A) $\mathcal{T}$ is one-to-one.

(B) $\mathcal{T}$ is onto.

(C) The range of $\mathcal{T}$ is a vector space over $\mathbb{R}$ with dimension equal to one.

(D) There does not exist $\underline{x} \in \mathbb{R}^2$ such that $\mathcal{T}(\underline{x}) = \begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix}$.

(E) None of the above is true.

11. Consider the following differential equation:

$$(2y^2 - 9xy)dx + (3xy - 6x^2)dy = 0.$$

Which of the following statements is/are true?

(A) This is a nonlinear differential equation.

(B) This equation is not exact.

(C) There exists an integrating factor that depends only on x.

(D) There exists an integrating factor that depends only on y.

(E) None of the above is true.

12. Continue from Problem 11. Determine the particular solution $y(x)$ satisfying $y(1) = 3$. Which of the following statements is/are true?

(A) $y(2) = 0$.

(B) $y'(4) = 3$.

(C) $y(3) = 9$.

(D) $y'(6) = 0$.

(E) None of the above is true.

13. Consider the following differential equation:

$$(x-4+4x^{-1})y''(x)+(2-x)y'(x)+y(x)=1.$$

Which of the following statements is/are true?

(A) This is a nonhomogeneous differential equation.

(B) Suppose $y_1(x)$ and $y_2(x)$ are two particular solutions of this equation. Then $y_1(x)+y_2(x)$ is also a solution of this equation.

(C) There exists a solution $y(x)=e^{\lambda x}$, where λ is a nonzero constant.

(D) There exist three particular solutions that are linearly independent over $\mathbb{R}$ for this equation.

(E) None of the above is true.

14. Continue from Problem 13. Determine the particular solution $y(x)$ satisfying $y(1)=0$ and $y''(3)=0$. Which of the following statements is/are true?

(A) $y(2)=4$.

(B) $y(4)=3$.

(C) $y'(6)=0$.

(D) $y'(8)=1$.

(E) None of the above is true.

15. Consider the following system of differential equations:

$$x'(t)+y'(t)+2y(t)=0,$$
$$x'(t)-3x(t)-2y(t)=0.$$

Suppose $x(0)=1$ and $y(0)=-3$. Which of the following statements is/are true?

(A) $x''(t)-x'(t)+6x(t)=0$.

(B) $y''(t)-2y'(t)+6y(t)=0$.

(C) $y''(0)=9x'(0)$.

(D) $x''(0)=y'(0)$.

(E) None of the above is true.

16. For the following second order differential equation

$$(2x^2 - x^3)y''(x) + (3x - 2x^2)y'(x) - (1 + 8x)y(x) = 0,$$

let $y_1(x) = x^{r_1} \sum_{n \geq 0} a_n x^n$ and $y_2(x) = x^{r_2} \sum_{n \geq 0} b_n x^n$ be the two linearly independent Frobenius series solutions for $y(x)$ when $x > 0$, where r_1 and r_2 are the zeros of the corresponding indicial equation with $r_1 \geq r_2$. Which of the following statements is/are true?

(A) $r_1 - r_2$ is not an integer.

(B) $r_1 + r_2 > 0$.

(C) $|r_1 r_2| = 4$.

(D) $r_1/r_2 > 0$.

(E) None of the above is true.

17. Consider the following integral equation

$$y(t) + 2\int_0^t y(\tau)\cos(t - \tau)\, d\tau = 3e^{-t} + 2\sin(t).$$

Which of the following statements is/are true regarding the values of $Y(s) = \mathcal{L}\{y(t)\}$?

(A) $Y(0) = 4$.

(B) $Y(-2) < 0$.

(C) $|Y(\sqrt{-1})| = 1$.

(D) $Y(1) > 1$.

(E) None of the above is true.

18. Continue from Problem 17. Which of the following statements is/are true regarding the solution $y(t)$ to the integrgal equation?

(A) $y(1) > 0$

(B) $y'(0) < 0$

(C) $y''(0) > 0$.

(D) $y(t) > 0$ for all $t \in \mathbb{R}$.

(E) None of the above is true.

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19. Consider the following boundary value problem for the bivariate function $f(x,t)$ that is defined for $x \in [0,2]$ and $t \geq 0$ and satisfies the following conditions

$$3\frac{\partial}{\partial t}f(x,t) = \frac{\partial^2}{\partial x^2}f(x,t),$$

$$\frac{\partial}{\partial x}f(x,t)\Big|_{x=0} = \frac{\partial}{\partial x}f(x,t)\Big|_{x=2} = 0, \quad \text{for all } t > 0,$$

$$f(x,0) = 4x.$$

The solution $f(x,t)$ can be represented in the following form

$$f(x,t) = \sum_{n=0}^{\infty} e^{d_n t}\left[a_n \cos(c_n x) + b_n \sin(c_n x)\right]$$

for some $a_n, b_n, c_n, d_n \in \mathbb{R}$ with $c_n \geq 0$, $d_0 > d_1 > \cdots$ and $|a_n| + |b_n| > 0$ for all $n = 0, 1, \ldots$ Which of the following statements is/are true regarding the values of a_n and b_n in the representation of solution $f(x,t)$?

(A) $a_0 \geq 0$ and $b_0 \leq 0$.

(B) $a_1 \leq 1$.

(C) $a_1/a_2 = 9$.

(D) $a_2 > b_2$.

(E) None of the above is true.

20. Continue from Problem 19. Which of the following statements is/are true regarding the values of c_n and d_n in the representation of solution $f(x,t)$?

(A) $d_0 = 0$.

(B) $c_1^2 + d_1 = \frac{\pi^2}{3}$.

(C) $c_2^2 + d_1 = -\frac{13\pi^2}{5}$.

(D) $c_3 = \frac{5\pi}{2}$.

(E) None of the above is true.

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