科目：高等微積分　類組別：C-2　共 1 頁第 1 頁　*請在試卷答案卷（卡）內作答

(1) Let $\{a_n\}$ be a sequence of positive real numbers. Prove that

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}. \quad (10\%)$$

(2) Let

$$\begin{aligned}
A &= \{(x,0) \in \mathbb{R}^2 \ : \ 0 < x < 1\},\\
B &= \{(x,y) \in \mathbb{R}^2 \ : \ 0 < xy < 1\},\\
C &= \{(x,y) \in \mathbb{R}^2 \ : \ x \text{ is rational and } 0 \le y \le 1\},\\
D &= \{(0,0)\} \cup \{(\frac{1}{x},0) \in \mathbb{R}^2 \ : \ x = 1,2,3,\cdots\},\\
E &= \{(3x+2y, 8x-9y) \in \mathbb{R}^2 \ : \ (x,y) \in \mathbb{R}^2, x^2+y^2 < 1\},\\
F &= \{(e^{\sin(xy)}, e^{\cos(x+y)}) \in \mathbb{R}^2 \ : \ (x,y) \in \mathbb{R}^2, x^2+y^2 \le 2\}.
\end{aligned}$$

(a) Which sets are open in $\mathbb{R}^2$? (Don't need to prove.) (4%)
(b) Which sets are compact in $\mathbb{R}^2$? (Don't need to prove.) (4%)
(c) Which sets are connected in $\mathbb{R}^2$? (Don't need to prove.) (4%)
(d) Which sets are complete in $\mathbb{R}^2$? (Don't need to prove.) (4%)

(3) Is the function $f(x) = \sqrt[3]{x^2}$ uniformly continuous on $\mathbb{R}$? Give your proof. (15%)

(4) Determine whether the function $f(x) = \sum_{n=0}^{\infty} \frac{x^{3n}\sin nx}{(n!)^3}$ is continuous on $\mathbb{R}$. Give your proof. (10%)

(5) Let $f : (-1,\infty) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{|x|}{4} & \text{if } x \text{ is rational and } -1 < x < \infty; \\ \dfrac{3x^2}{9+4x^2} & \text{if } x \text{ is irrational and } -1 < x < \infty. \end{cases}$$

Prove that there exists one and only one point $c \in (-1,\infty)$ such that $f'(c)$ exists, and, find c and $f'(c)$. (15%)

(6) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by setting $f(0,0) = 0$ and

$$f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2} \quad \text{if} \quad (x,y) \ne (0,0).$$

(a) Do $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$? Give your proof. (4%)
(b) Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(x,y) \ne (0,0)$. (4%)
(c) Is f of class C^1 on $\mathbb{R}^2$? Give your proof. (10%)
(d) Is f of class C^2 on $\mathbb{R}^2$? Give your proof. (6%)
(Recall that a function is said to be of class C^r if the first r derivatives exist and are continuous.)

(7) Let $f : [0,2] \to \mathbb{R}$ be Riemann integrable on $[0,2]$. Assume that the set $A = \{x \in [0,2] : f(x) = 2\}$ is dense in $[0,2]$. Find the integral $\int_0^2 f(x)dx$. (10%)