台灣聯合大學系統94學年度學士班轉學生考試命題紙

科目：高等微積分　　類組別：D2　　共 1 頁第 1 頁　　*請在試卷答案卷（卡）內作答

(1) Let $\{a_n\}$ and $\{b_n\}$ be two bounded sequences in $\mathbb{R}$. If for every a_n there is some $k \geq n$ such that $b_k \geq a_n$, prove that

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n. \qquad (10\%)$$

(2) Let X be the metric space of irrational numbers with the metric $d(x,y) = |x-y|$. Let A be the set of all points x in X with $3 \leq x^2 < 9$. Answer the following questions. In all cases, give your proofs.

(a) Is A closed in X? (5%)

(b) Is A open in X? (5%)

(c) Is A compact? (5%)

(d) Is A connected? (5%)

(3) Let (X,d) and (Y,ρ) be metric spaces and $f: X \to Y$ be a continuous function on X. Prove that $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$. Here $\overline{A}$ denotes the closure of A. (10%)

(4) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by setting $f(0,0) = 0$ and

$$f(x,y) = \frac{y^3}{x^2+y^2} \quad \text{if} \quad (x,y) \neq (0,0).$$

(a) Do $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$? Evaluate it when it exists. (5%)

(b) Is f continuous at $(0,0)$? Justify your answer. (5%)

(c) Is f differentiable at $(0,0)$? Justify your answer. (10%)

(5) Let $f: [0,1] \to \mathbb{R}$ be such that

$$f(x) = \begin{cases} 0 & \text{if } x = \frac{n-1}{n},\ n = 1,2,3,\cdots, \\ 1 & \text{otherwise.} \end{cases}$$

Prove that f is integrable on $[0,1]$ and find the value of $\int_0^1 f(x)dx$. (10%)

(6) Determine whether the sequence of functions $f_n(x) = \frac{x}{1+nx^2}$, $n = 1,2,3,\ldots$, converges uniformly on $\mathbb{R}$ as $n \to \infty$. Give your proof. (10%)

(7) Let $\{a_n\}$ be a bounded sequence in $\mathbb{R}$. Determine whether the function $f(x) = \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n$ is continuous on $\mathbb{R}$. Give your proof. (10%)

(8) Prove that the equation

$$y\cos x = x^2 - e^x \cos y$$

has a solution of the form $y = g(x)$ for (x,y) near $(0,0)$. Find the first three terms in the Taylor expansion of $g(x)$ about $x = 0$. (10%)