科目 高等微積分 類組別 032 共 1 頁第 1 頁 *請在試卷答案卷（卡）內作答

(1) (20%) In the following each statement, is it true or false ? If it is true, prove it. If your answer is false, give an example.

(a) If x_n satisfies $|x_{n+1} - x_n| < \frac{1}{5^n}$, then x_n converges.

(b) If x_n satisfies $|x_{n+1} - x_n| < \frac{1}{\sqrt{n}}$, then x_n is convergent.

(c) If x_n is a monotone increasing sequence such that $x_{n+1} - x_n \le \frac{1}{n}$, then x_n converges.

(2) (20%) Let $Q^c = R \backslash Q$, where Q is the set of all rational numbers.

(a) Is $Q^c \cap [0,1]$ a *compact set* ? Prove your answer.

(b) Let

$$f(x) = \begin{cases} x, & \text{if } x \in Q^c; \\ p \sin \frac{1}{q} & \text{if } x = \frac{p}{q}, \end{cases}$$

where p and q are integers such that $(p,q) = 1$. At what points is f continuous ?

(c) Let

$$g(x) = \begin{cases} 0 \text{ if } x \in Q^c \\ \frac{1}{q} \text{ if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

Prove that f is continuous on Q^c and discontinuous otherwise.

(3) (20%) Compute the following limits.

(a) $\lim\limits_{n\to\infty} \sum_{k=1}^{n} (n^2 + k^2)^{\frac{-1}{2}}$;

(b) $\lim\limits_{n\to\infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}}$, where $p > 0$.

(4) (20%) Let $F(\mathbf{x}, y)$ be a function of class C^1 on some neighborhood of a point $(\mathbf{a}, b) \in R^n \times R$. Suppose that $F(\mathbf{a}, b) = 0$ and the partial derivative $\partial_y F(\mathbf{a}, b) \neq 0$. Prove that there exist positive numbers r_0, r_1 such that the following conclusions (A) and (B) are valid:

(A) For each $\mathbf{x}$ in the ball $||\mathbf{x} - \mathbf{a}|| < r_0$ there is a unique y such that $|y - b| < r_1$ and $F(\mathbf{x}, y) = 0$. We denote this y by $f(\mathbf{x})$; in particular, $f(\mathbf{a}) = b$.

(B) The function f thus defined for $||\mathbf{x} - \mathbf{a}|| < r_0$ is of class C^1, and its partial derivatives are given by

$$\partial_j f(\mathbf{x}) = -\frac{\partial_j F(\mathbf{x}, f(\mathbf{x}))}{\partial_y F(\mathbf{x}, f(\mathbf{x}))}, j = 1, 2, \cdots, n.$$

(C) Suppose $F(x, y)$ is a C^1 function such that $F(0,0) = 0$. What conditions on F will guarantee that the equation $F(F(x,y), y) = 0$ can be solved for y as a C^1 function of x near $(0,0)$?

(5) (20%) Let $f_n(x) = xe^{-nx}, x \in [0, \infty), n = 0, 1, 2, \ldots$.

(a) Show that $f(x) = \sum\limits_{n=0}^{\infty} f_n(x)$ exists. Compute f explicitly.

(b) Is f continuous ?

(c) Find a suitable set on which the convergence is uniform.

(d) May we differentiate term by term on $(0, \infty)$? Why ?