科目 應用數學 類組別 037 共 2 頁第 1 頁 *請在試卷答案卷（卡）內作答

1. A damping motion of a unit mass on a spring, $y(t)$, is governed by a second-order differential equation $\ddot{y}+c\dot{y}+ky=0$, where c is the damping factor and k is the spring constant, and both are constants.

(a) Please convert this 2nd-order differential equation to a 1st-order differential system. Write down the system in a matrix form. (10%)

(b) Assuming that $c = 6$ and $k = 8$, please show two eigenvalues are -2 and -4 for the solution of this vibrating system. (5%)

(c) Assuming that $c = 0$ and $k = 4$, please show two eigenvalues are $2i$ and $-2i$ for the system ($i=\sqrt{-1}$). (5%)

(d) Assuming that $c = 2$ and $k = 2$, please show two eigenvalues are $-1\pm i$ for the motion. (5%)

(e) Considering the real and imaginary parts of these eigenvalues, please briefly explain the types of critical points in the phase plane, i.e. the $y\dot{y}$-plane, for the above three cases are (b) improper node (Fig. 1), (c) center (Fig. 2) and (d) spiral point (Fig. 3), respectively. (15%)

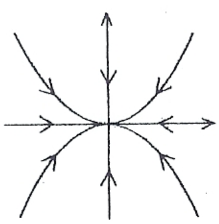

Figure 1

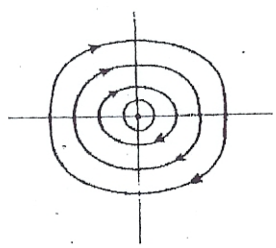

Figure 2

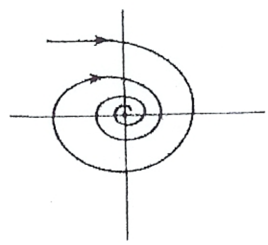

Figure 3

2. For an infinite bar, one-dimensional heat equation $\frac{\partial u(x,t)}{\partial t}=c^2\frac{\partial^2 u(x,t)}{\partial x^2}$ with the initial condition $u(x,0)=f(x)$ has an solution of the error function form

$$u(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2cz\sqrt{t})e^{-z^2}\,dz\,.$$

(a) Please derive the solution shown above by Fourier integrals. (20%)

(b) If $f(x)=1$ when $x>0$ and $f(x)=0$ when $x<0$, please show that the solution

$$u(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\frac{x}{2c\sqrt{t}}}^{\infty} e^{-z^2}\,dz \quad \text{for } t>0. \ (10\%)$$

3. For a non-homogeneous differential equation $y''+p(x)y'+q(x)y=r(x)$, please derive the following solution by the method of variation of parameters

$$y_p(x)=-y_1\int\frac{y_2 r}{y_1y_2'-y_2y_1'}dx+y_2\int\frac{y_1 r}{y_1y_2'-y_2y_1'}dx$$

where y_1, y_2 form a basis of solutions of the homogeneous equation $y''+p(x)y'+q(x)y=0$. (20%)

4. Assuming sufficient differentiability, please show that:

(a) $\nabla\times(\nabla f)=\vec{0}$, here f is a scalar function. (5%)

(b) $\nabla\cdot(\vec{u}\times\vec{v})=\vec{v}\cdot\nabla\times\vec{u}-\vec{u}\cdot\nabla\times\vec{v}$. (5%)